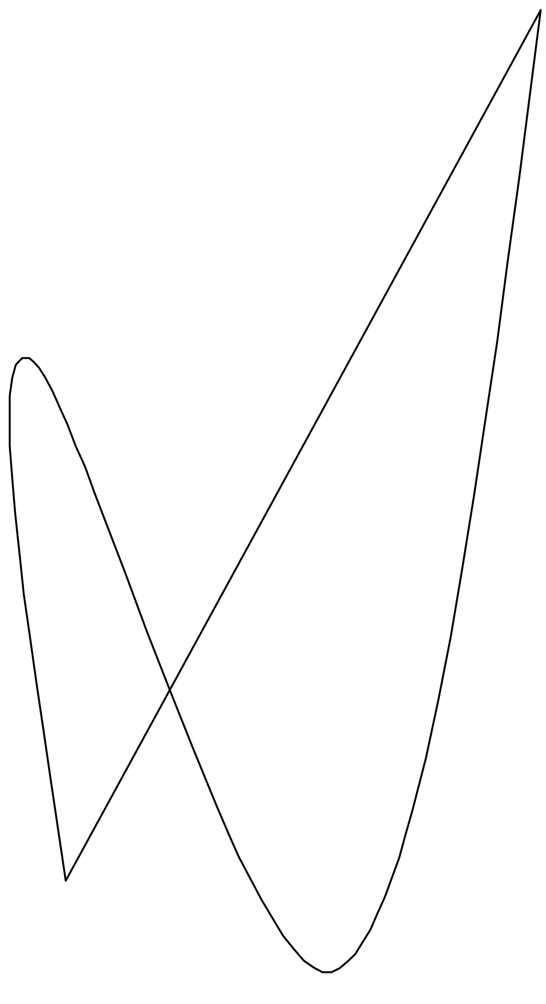
University of Bergen
Department of Economics

Economic models of water-resources



Student;
Vífill Karlsson

Thesis advisor;
Professor, Eirik Schröder Amundsen

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Chapter 1

1 Introduction.

Water has been scarce in many regions of the world. In papers written on this issue until now, economists seem to been ignoring the role of the resource size on its growth. Accordingly its significance is the main objective of this paper. Thoroughly well known relations from the hydrogeology support this significance. The relations will be helpful in developing two functions, one for each society, that is constraints of a model. This model appears frequently in papers on resource economics, the optimal control theory. Then solving the model to find the optimal solution for the society, describe the dynamics and a suggestion for the best management.

There are two types of water systems under discussion. These are a groundwater system and a reservoir system. They are representative for the water systems of two communities of northern Europe. These communities seem to have more than enough freshwater for domestic demand. The main objective of these communities' water suppliers is to provide the inhabitants enough of "good" freshwater, at a low cost. The second objective of this paper is to compare these communities' optimal solutions.

Chapter 2 handles arguments for the role of the resource size on its growth. The chapter is divided in two main sections where it distinguishes between a ground water- and a reservoir system. Two growth models, one for each water-system, are educted. So the results of chapter two are two dynamic constraints containing e.g. these growth models.

Chapter 3 concentrates on the model. The model is constructed applying the optimal control theory where the results from chapter 2 become its main constraint. Then the optimal solution for each society is found. The solutions do it easy to make the same general description and suggestion for the best management. Finally, the societies compared by emphasising their main differences.

Chapter 2

2 General characters of the water- resources.

2.1 Introduction.

There are two types of water-resources under discussion in this paper, a ground water system and a reservoir system. The intention of chapter 2 is to describe the characteristic of its growth. To do that we emphasise the role of the resource size on its growth. The size of the resource does influence its growth mainly in two ways. These are its ability to collect water and keep it (fresh and pure). These relations are well known in the hydrogeology but have rarely been stressed in resource economics. The objective of chapter 2 is to find a relevant equation for the resource's growth. Expand it to full-developed dynamic constraint usable in the context of optimal control theory for chapter 3. So there are two problems in this chapter to solve:

Problem 1: *What is the dynamic constraint to the social maximum problem that describes the growth of a groundwater system?*

Problem 2: *What is the dynamic constraint to the social maximum problem that describes the growth of a reservoir system?*

The theoretical facts in this chapter are generally based on Fetter (1980) if nothing else is mentioned.

2.2 Hydrologic cycle

Water flows continually in a circle that is known as a *Hydrologic Cycle* (Fetter, 1980, pp. 4-6). It is described in both in figure 2.1 and 2.2. The ocean is the first link to be mentioned in the hydrologic cycle. It contains about 97,2 per cent of the Earth's total water stock. Water evaporates from the surface of the ocean. The amount of evaporated water varies between locations. This depends on the solar radiation that leads us to greatest evaporation around the equator in the wide world. After a while, water vapour condenses and forms droplets. Part of these droplets falls on land. There they enter two main types of pathways, the *overland flow* and the *infiltration*. If the surface soil is tight enough it is more likely that these droplets will enter overland flow like rivers and lakes. If the surface soil is porous enough they enter infiltration. Some of the water in the infiltration pathway is used by plants and then transpired as vapour into the atmosphere. The rest of the water in the infiltration pathway continues to fall until it reaches a rock or soil that is saturated with water. The highest level of the saturated area is the *water table* and below that is the *ground water*. Even though a particular drop enters one of these pathways in the beginning it could later on go from overland flow to infiltration or vice versa; depend on, for example the landscape.

Figure 2.1

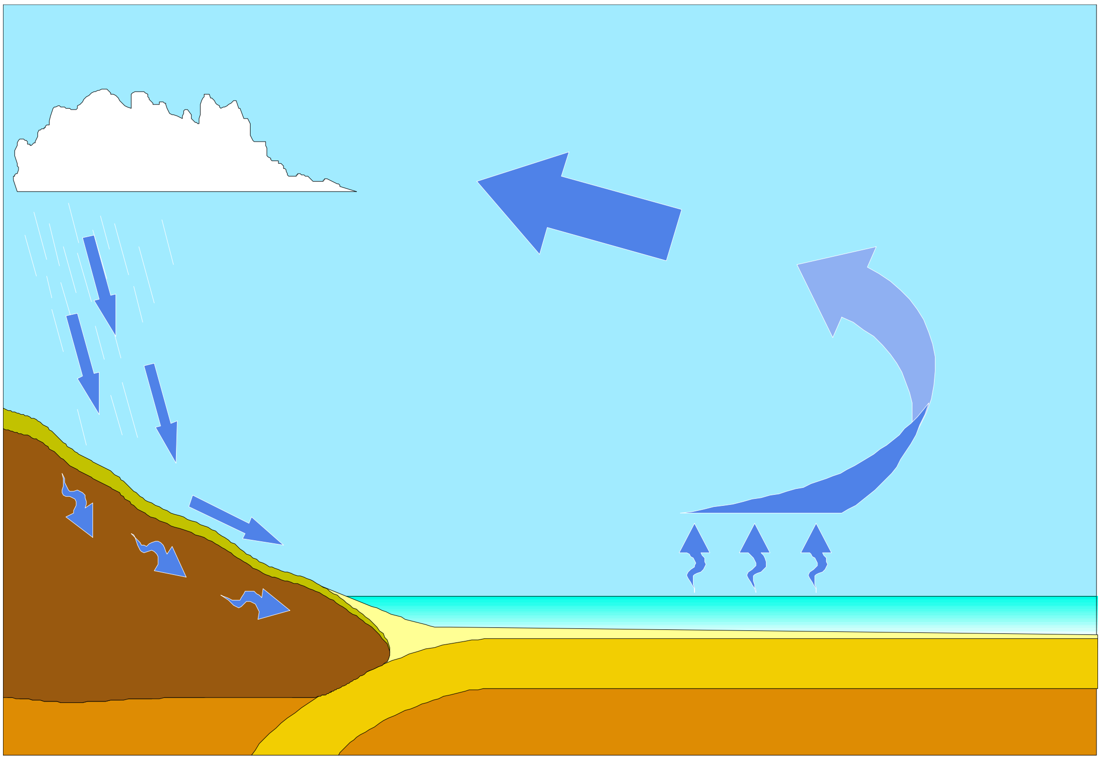

Figure 2.1 is the hydrologic cycle under natural conditions. Water evaporates from the surface of the ocean. After a while, water vapour condenses and forms droplets. Part of these droplets falls on land. There they enter two main types of pathways, the Overland flow and the infiltration. By the law of gravity the water will reach the ocean again.

The hydrologic equation is very helpful in the case of quantitative study as mentioned by Fetter (1980, p. 7).

Recharge = Discharge ± Changes in storage

The equation above can be applied to any hydrologic system. Every hydrologic system contains a water-storage that is the accumulated difference between the recharge and the discharge. The equation is time-dependent. At any time the difference between the inflow rate and the outflow rate is the rate of changes in storage which can be either positive or negative.

Figure 2.2

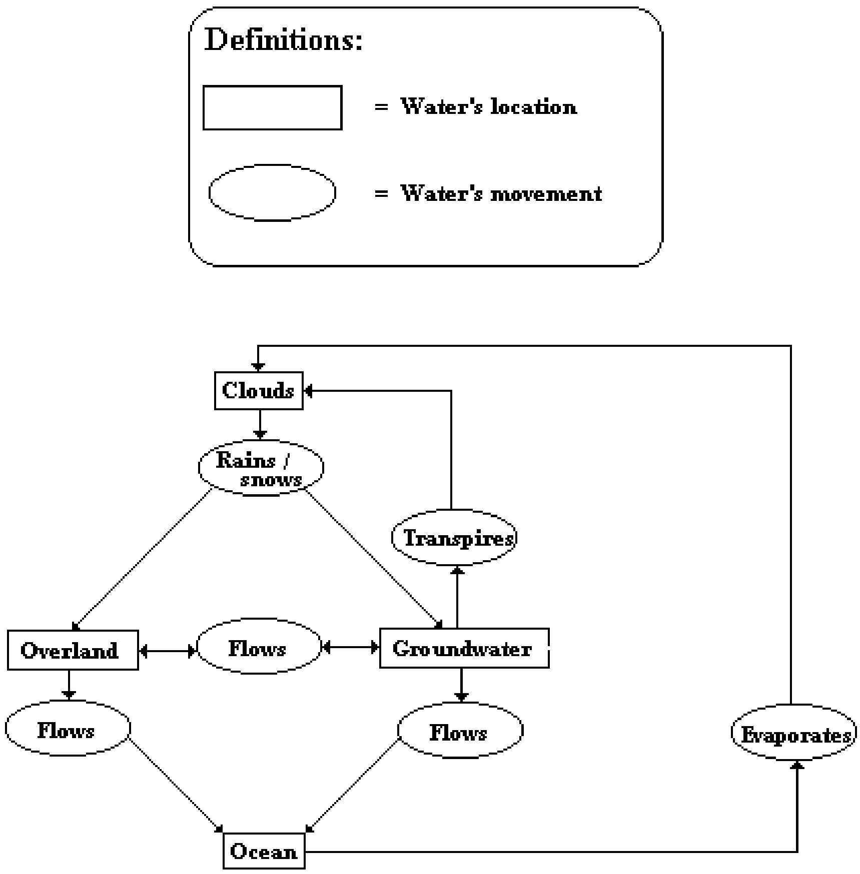

Figure 2.2 is the hydrologic cycle under natural conditions. Water evaporates from the surface of the ocean. After a while, water vapour condenses and forms droplets. Part of these droplets falls on land. There they enter two main types of pathways, the overland flow and the infiltration. Some of the water in the infiltration pathway is used by plants and then transpired as vapour into the atmosphere. The rest of the water in the infiltration pathway continues to fall until it reaches a rock or soil that is saturated with water. The highest level of the saturated area is the water table and below that is the ground water. Even though a particular drop enters one of these pathways in the beginning it could later on go from overland flow to infiltration or vice versa.

2.3 Precipitation

"In most parts of the world, the majority of precipitation falls as rain..." (Brassington,

1988, p. 93). This is why precipitation is considered as rain in this paper. Snow is considered as delayed rain.

Rainfall is highly stochastic and fluctuating. Havenner and Tracy mentioned (1992, pp. 171-172) that stream flow has daily and particularly seasonal fluctuations, caused by fluctuating rainfall. One has certain ideas about these fluctuations, based on "historical" data and other research but nothing is sure about it, even within a day.

What is the geological origin of the entering water? Where does it fall into as rain? Some of it fell directly into a well. Some of it probably flowed through many places before it got into a well, like precipitation that fell on glaciers. It is easy to split both Iceland and Norway into very many *rainfall zones*[1]. Therefore it is possible for ground water to receive water from various rainfall zones. The probability of precipitation is very different both between rainfall zones and seasons. The differences between the North- and Southwest of Iceland and the West and East of Norway are good examples.

Plants instantly catch a part of the rain. The other part of it reaches the ground. As mentioned before, that part will either enter infiltration or overland flow. It will enter overland flow only if the precipitation rate gets greater than grounds maximum infiltration rate. The infiltrated waters that pass evaporation and transpiration will eventually join the ground water.

It is already obvious that the relationship between rainfall and the growth of ground water-resource is very complicated. The main characters could be demonstrated by the following equation despite its complexity:

$$Z_t = \sum_{t=1}^{k} \sum_{i=1}^{j} (\pi r^2)_1 \, p_{t1} e_{t1} y_{t1} + \ldots + (\pi r^2)_j \, p_{tj} e_{tj} y_{tj}$$

[1] Definition: *Rainfall zone* is a zone, which could be described with approximately the same precipitation characters.

Where

Z_t	=	Waterquantity that becomes ground water in period *t*.
y_{ti}	=	Average quantity of rain in period *t* in the rain zone *i* .
p_{ti}	=	Probability of precipitation in period *t* in the rain zone *i*.
e_{ti}	=	Evapo-transpiration and runoffs (proportion) in period *t* in rain zone *i*.
$(\pi r^2)_i$	=	Size of the *i*-rainfall zone.

As claimed by Árnason (1976, pp. 9-11) all ground water in Iceland, hot and cold, has a meteoric origin. In Reykjavik, the connection between the local rainfall zone and its groundwater level is so tight that it does not create any problems to dismiss any effects from other rainfall zones. Even though other rainfall zones influence is neglected, the problem is still that some proportion of the rainfall is entering instantly and some of it later. However each day's rainfall has its maximum period of influence. In Reykjavik, this period is assumed to last for 24 hours, 60 days after each particular rainy day. In other words; rainfall of quantity y_{t-n} will give quantity of water that becomes ground water equal to Z_t, where n is equal to 2 in this context. By neglecting possible influence from other rainfall zones the following equation can be relevant in describing the relationship between rainfall and arriving volume of water,

$$Z_t = Z_t(y_{t-n}) = (\pi r^2)e_{t-n}p_{t-n}y_{t-n}$$

Since the rain is known, the probability $p_{t-n} = 1$ and the equation changes to,

$$Z_t = Z_t(y_{t-n}) = (\pi r^2)e_{t-n}y_{t-n}$$

It is also demonstrated by Huisman (1972, p. 97). The colder the part of the world the smaller the evaporation. The fewer trees that are in the area the lesser is the transpiration. Therefore it is appropriate to assume negligible transpiration and evaporation in a barren land of ice, as Iceland. In this case, very high proportion of the infiltrated water will eventually enter ground water. Then $e_{t-n} = 1$ and equation ($E2.1$) drops out from the expression above,

$$(E2.1) \qquad Z_t = Z_t(y_{t-n}) = \pi r^2 y_{t-n}$$

$$=> (E2.1.1) \qquad r^2 = \frac{Z_t}{\pi y_{t-n}}$$

The equation (E2.1.1) shows how big the area that is withdrawn (y ↑ then r ↓, Z ↑ then r ↑). Equation (E2.1.1) shows how much water it will be possible to pump from the aquifer (r ↑ then Z*(y)* ↑, y ↑ thus Z*(y)* ↑). Here z is the available water.

Equation (E2.1) shows the quantity of water that arrives ground water as a function of precipitation, y, and the size of the rainfall zone, πr^2. It claims that the quantity of water that reaches ground water is function of precipitation multiplied by the size of the area (rainfall zone).

Eventually some fraction of Z_t (the Quantity of water that reaches ground water) reaches the *saturated zone*[2] of the ground water, but the saturated zone is not a static storage. If the rainfall continues the water level in the saturated zone will rise and flow towards lower location. Thereafter the water level will reach the lowest surface in the area, flow out, create a lake and probably a creek. This leads to a slope of the water level in the saturated zone, with the highest level in the recharging area and the lowest one in the discharging area. If the rainfall gets more intensive, the water level in the saturated zone will increase, and so will the pressure; thus the flow of groundwater.

2.4 Ground water system

The following description of the ground water system is a combined description of the theoretical ground water system and the actual ground water system. It is a source to the supplied water in Reykjavik, the capital of Iceland, which has a population of about 120.000 inhabitants.

2.4.1 Ground quality

It is one thing to have the rain on the ground. Another thing to have a ground that has

[2] Definition: *Saturated zone* is that part of the soil that is saturated with water.

all the necessary and *sufficient conditions*[3] to absorb the water and create an *efficient saturated zone*[4]. Water cannot infiltrate a frozen ground because it is tight as a concrete pavement, leaving for example the South Pole devoid of any ground water. What are these necessary and sufficient conditions?

Porosity

For the land surface to be able to absorb the rain it has to have openings. The ground is a "collection" of various earth materials like rocks and sediments. Sediments are assemblages of individual grains with openings between them. Some rocks contain void spaces; others are solid. The ground is continually under formation by physical and chemical weathering processes, thus creates and changes voids. Voids are a *necessary condition*[5] for droplets to infiltrate the ground and the void skeletons are very different between earth materials. This is a matter of *porosity*[6]. Porosity is defined mathematically by the equation (Fetter, 1980, p. 63):

$$n = \frac{100V_v}{V}$$

Where

n	=	The porosity (percentage).
V_v	=	The volume of void space in a unit volume of earth material.
V	=	The unit volume of earth material, including both voids and solids.

A volume of rock is saturated with water. Suppose that the discharge is greater than the recharge in a particular period. Then some fraction of the water will flow downward because of the gravity. This fraction is called *specific yield*[7] (S_y). The other fraction of the water remains around the mineral grain because of surface tension. That

[3] Definition: *Sufficient condition:* Statement *p* is true when *q* is true, but *p* can also be true when *q* is not true. In this case, *q* is said to be a sufficient condition for *p*.
[4] Definition: An *efficient saturated zone* is a saturated zone that has the ability to support a well.
[5] Definition: "A *Necessary condition* is in the nature of a prerequisite: suppose that a statement *p* is true only if another statement *q* is true; then *q* constitutes a necessary condition of *p* (Chiang, 1984, p. 89).
[6] Definition: "*The porosity of earth materials is the percentage of the rock or soil that is void of material*." (Fetter, 1980, p. 63).
[7] Definition: "*Specific yield is the ratio of the volume of water that drains from a saturated rock owing to the attraction of gravity to the total volume of the rock*" (Fetter, 1980, p.73).

fraction is called *specific retention*[8] (S_r). The sum of these two fractions gives us the total porosity defined by the following mathematical equation (Fetter, 1980, p. 74):

$$n = S_y + S_r$$

Where

S_y	=	Specific yield
S_r	=	Specific retention

Porosity is necessary but is not a sufficient condition for the ground's ability to create an efficient saturated zone. To realise this let us imagine that a volume, called volume 1, filled with small grains and another volume, called volume 2, filled with bigger grains. Volume 1 and 2 is assumed to have the same size in cubic meters. The total surface of the grains in volume 1 is greater than in volume 2, meaning that the total S_r is higher in volume of smaller grains. Clay can have a value of S_r equal to 48%, which generally is approved as sufficient porosity. However the voids are very narrow. Therefore an efficient saturated zone has to be able to refill a well from its storage of water and receive new water from precipitation. Will ground of very narrow voids be able to do that?

Hydraulic conductivity

Refilling the well is a very significant ability. It can be very hard to build a well, which usually is a deep and narrow cavity. The exact volume of this hole is a very small proportion of the expected supply of water from it. So, it is a great advantage to have even and sufficient recharge of a well. To have this preferable condition it is necessary to have sufficient rainfall and flow of ground water, or *hydraulic conductivity*[9]. Hydraulic conductivity can be characterised by the following equation (Fetter, 1980, p. 78):

$$K = (L/t)$$

8 Definition: *"The specific retention (S_r) of a rock or soil is the ratio of the volume of water a rock can retain against gravity drainage to the total volume of the rock"* (Fetter, 1980, p. 73).

9 Definition: *Hydraulic conductivity* (K) is "*A coefficient of proportionality describing the rate at which water can move through a permeable medium. The density and kinematic viscosity of the water must be considered in determining hydraulic conductivity*" (Fetter, 1980, p. 571).

Where

K	=	Hydraulic conductivity
L	=	Length
t	=	Time

K is a function of length and time, or velocity. The function can be rearranged to (Fetter, 1980, p. 78):

$$K = Cd^2 \frac{\rho_w g}{\mu}$$

Where

C	=	Shape factor
d	=	The square of the mean pore diameter
g	=	Acceleration of gravity
ρ_W	=	Density of the water
μ	=	Dynamic viscosity
$\rho_W g$	=	Specific weight of the water

There are many things that influence the hydraulic conductivity. The hydraulic conductivity (K) depends on the fluids specific weight ($\rho_W g$). The hydraulic conductivity, K, increases with higher specific weight. The dynamic viscosity (μ) measures the resistance of the fluid to the shearing, which is necessary for the fluid flow. The flow increases with lower dynamic viscosity. The fluid's temperature influences both the dynamic viscosity, μ, and the density, ρ_W. Colder fluid gives higher μ (more viscous). However the relationship between temperature and density (ρ_W) is more complex. When water's temperature is higher than 4°C, the fall of it will reduce the density. Therefore it can be claimed that if the water temperature is higher than 4°C the fall of it will also reduce the hydraulic conductivity (K).

Cd^2 is the influence from the porous medium on K. As said before, the smaller mineral grains, the smaller the openings between them (smaller d^2). This leads to slower flow of water through the sediments, or lower hydraulic conductivity (K). The shape of the openings will also influence the hydraulic conductivity, K. It is easier for the water to travel through straight corridors than crooked ones. This indicates for example a different degree of impedance. C is a constant that will collect overall effect of the shape of the voids. Cd^2 is known as the concept *intrinsic permeability*[10]. Intrinsic permeability of a certain size (usually greater than 0,01) is a necessary and a sufficient condition for the ground's ability to create an efficient saturated zone, which generally is called an *aquifer*[11].

If the sediment openings are extremely wide, the flow groundwater will be much faster. It can be undesirable because the ability to retain water is also preferable. Then higher average precipitation rate can be desirable to create and maintain an aquifer. So hydraulic conductivity has both positive and negative aspects.

2.4.2 Aquifers

How will porosity and hydraulic conductivity reflect in the qualities of an aquifer? Putting these concepts into a context of aquifers, the same basic questions of interests are raised: How much water and how fast can the aquifer pass through itself?

Transmissivity

Under natural conditions, an aquifer is often in a state of dynamic equilibrium. Then the amount of water recharging the aquifer is equal to the amount discharging the aquifer. The *potentiometric surface*[12] is stable and the amount of water in storage is constant. The aquifer moves the water from the recharge zones to the discharge areas. The maximum amount of water that any section of the aquifer can transmit is a function of the *transmissivity*[13] and the maximum gradient of the potentiometric surface. The transmissivity can be mathematically stressed like Fetter (1980, p. 105) did:

[10] Definition: *"Intrinsic permeability pertaining to the relative ease with which a porous medium can transmit a liquid under a hydraulic or potential gradient. It is a property of the porous medium and is independent of the nature of the liquid or the potential field"* (Fetter, 1980, p. 572).

[11] Definition: *Aquifer* is "*rock or sediment in a formation, group of formations, or part of a formation which is saturated and sufficiently permeable to transmit economic quantities of water to wells and springs.*" (Fetter, 1980, p. 565).

[12] Definition: *Potentiometric surface* is *"a surface that represents the level to which water will rise in tightly cased wells. If the head varies significantly with depth in the aquifer, then there may be more than one potentiometric surface. The water table is a particular potentiometric surface for an unconfined aquifer."* (Fetter, 1980, p. 575).

[13] Definition: Aquifers *transmissivity* is a *"measure of the amount of water that can be transmitted horizontally by the full saturated thickness of the aquifer under a hydraulic gradient of 1."* (Fetter, 1980, p. 105).

$T = bK$

Where

T	=	The aquifer transmissivity
b	=	The saturated thickness of the aquifer

If the water table is far below the surface, the aquifer will not transmit water at full capacity. If however, the water table is close to the surface of an unconfined aquifer, the aquifer will be full and will transmit water at maximum capacity. Thus higher head increases transmissivity.

Storativity

The ability to store water is very significant in an aquifer. In a state of dynamic equilibrium the storage is stable. To understand the ability, the concept *specific storage*[14] is introduced. This concept can be used to both for confined and unconfined aquifers. The concept can be characterised mathematically as Fetter (1980, p. 106) did by the following equation:

$$S_S = \rho_W g(\alpha + n\beta)$$

Where

α	= The compressibility of the aquifer skeleton
β	= The compressibility of the water

The differences in the water table do influence the specific storage in the aquifer. The pressure in the aquifer increases when its head increases. Thus the mineral skeleton will expand which means, the space between the mineral grains' expands (see figure 2.3, cube A changes to cube B). The water contracts because the water's density increased (ρ_W). Therefore the *effective porosity*[15] also increases, meaning that many of

[14] Definition: "*Specific storage (S_S) is the amount of water per unit volume of a saturated formation that is stored or expelled from storage owing to compressibility of the mineral skeleton and the pore water per unit change in head.*" (Fetter, 1980, pp. 105-106).

[15] Definition: *Effective porosity Fluids flowing through the pores in rock and sediment may not move through the entire volume of pore space. There may be both noninterconnected pores as well as dead-end pores, which have no outlet so that water cannot flow through them. The effective porosity, n_e, is the porosity available for fluid flow. The effective pore fraction, epf, is the ratio of the porosity available to flow to the total porosity: $n_e = n * epf$"* (Fetter, 1980, p. 84).

the previously closed voids would open up and extremely narrow voids would be filled with contracted water. Finally, this process increases the specific storage (S_S). The opposite happens when the head declines (see figure 2.3, cube B changes to cube A). Pressure in the saturated aquifer decreases and the specific storage *(S_S)* decreases.

Figure 2.3

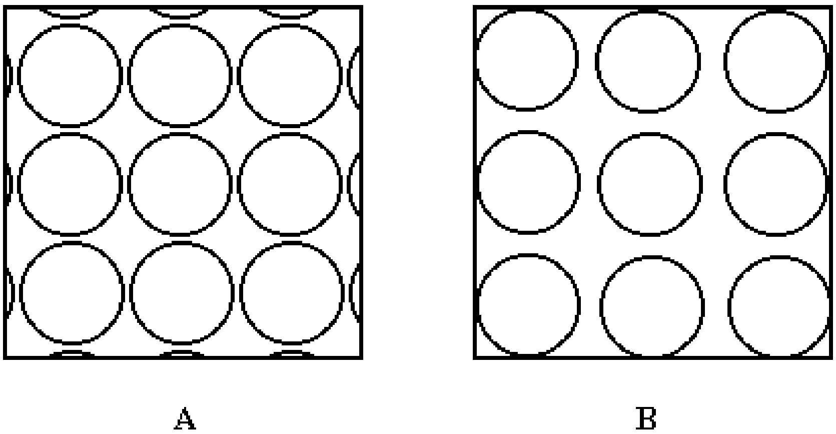

Figure 2.3 shows two cross sections from the same volume of a saturated ground. The circles are the mineral grains. The height of the saturated ground explains the difference between volume A and B. It is higher in volume B than in volume A. That is the reason for different space between the grains in the volumes. It is more space between the grains in volume B than in volume A. So the higher water tables the greater is the space between the grains.

In a <u>confined aquifer</u> a *storativity*[16] (S) is the product of the specific storage (S_S) and the aquifer thickness (b). It is defined mathematically like Fetter (1980, p. 107) did:

$$S = bS_S$$

Where

S = Storativity

In an <u>unconfined aquifer</u> the storativity (S) is found by the following equation (Fetter, 1980, p.107):

[16] Definition: "*Storativity (S), is the volume of water that a permeable unit will absorb or expel from storage per unit surface area per unit change in head.*" (Fetter, 1980, p. 105).

$$S = S_y + bS_s$$

S_y is added because it is the part of the unsaturated zone that will reach the saturated zone eventually. Clearly, there is a positive relationship between a specific storage and the storativity in both types of aquifers. An increased specific storage would lead to an increased storativity. The conclusion is that higher head increases the specific storage and specific storage increases storativity. **Thus higher head increases storativity**.

The aquifer under discussion is assumed to be *homogenous*[17] and *isotropic*[18]. The assumption of homogeneity gives us values of the transmissivity and storativity of the unit would be about the same wherever in the aquifer. The isotropic assumption gives us equal values of the transmissivity and storativity in any direction of flow.

2.4.3 Growth model for ground water system.

The growth of the aquifer

There are several external factors (incidents) which can move the aquifer from the state of dynamic equilibrium for example a heavy rainstorm. When the rate of recharge exceeds the rate of discharge, the aquifer grows. Thus the volume of water in the aquifer increases and the head rises. After some time the discharge rate has reached the rate of recharge and the aquifer reaches dynamic equilibrium again. Then it contains more volume of water, which flows through the aquifer with higher velocity than before. To make things easier, we consider the aquifer to be a control volume (figure 2.4 (Fetter, 1980, p. 132)). These are known assumptions and are often used in similar studies (Fetter, 1980, p.131).

[17] Definition: *"A homogenous unit is one that has the same properties at all locations."* (Fetter, 1980, p. 108).

[18] Definition: *Isotropy* is *"the condition which hydraulic properties of the aquifer are equal in all directions."* (Fetter, 1980, p. 572).

Figure 2.4

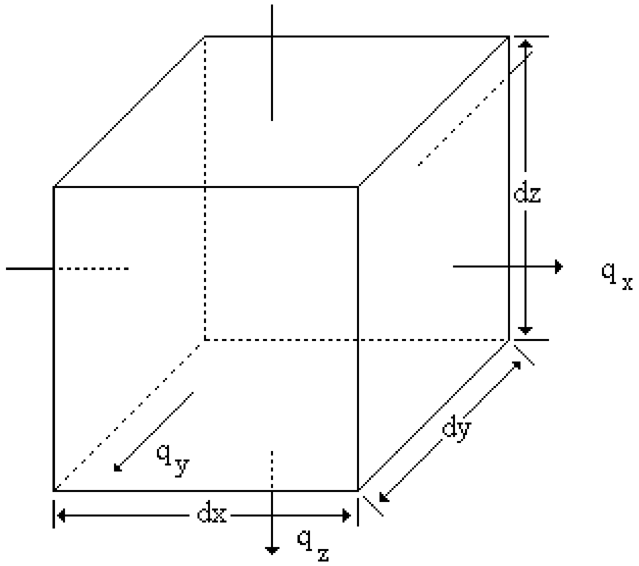

Figure 2.4 shows the control volume. The control volume is a cube from the aquifer. To keep the discussion simple the aquifer is assumed to be of this shape.

The water can flow through the control volume in three main directions (q_x, q_z and q_y in figure 2.4). Any change in the mass of water, m, with respect to time is rearranged by Fetter (1980, p. 133) to

$$\frac{\partial m}{\partial t} = (\alpha\rho_w g + n\beta\rho_w g)\rho_w dxdydz \frac{\partial h}{\partial t}$$

Where

m = Volume of water in the aquifer.
h = Potentiometric surface (head) of the aquifer.

Fetter (1980, p. 132) points out the following context

$$(C.1) \quad m = (\alpha\rho_w g + n\beta\rho_w g)\rho_w dxdydz$$

If $(\alpha\rho_w g + n\beta\rho_w g)\rho_w$ is defined equal to φ then context (C.1) turns to

$$(C.1) \quad m = \varphi dxdydz$$

Notice that φ must be greater than or equal to zero and less than or equal to one, 0 < φ < 1, because of the *dxdydz* is the size of the control volume (m^3) which determines the maximum size of the aquifer under discussion. The other part of "m", φ, is a fraction of that volume that is water, and the other fraction (1-φ) is solid. According to (*C*.1) following differential equation must occur:

$$(E2.2) \qquad \frac{\partial m}{\partial t} = m\frac{\partial h}{\partial t}$$

The potentiometric surface, *h*, is totally dependent to the amount of recharged water. *"Due to its direct contact with the atmosphere, an unconfined aquifer is commonly recharged by rainfall minus the losses of the evapo-transpiration. This recharge will result in a flow of groundwater to a pumped well."* (Huisman, 1972, p. 96). Fetter (1980, p. 444) agree with this but gets more specific by e.g. emphasise the role of infiltration. He claims that the amount of recharged water is influenced by the amount of infiltrated water that passes the evaporation and the transpiration. The amount of infiltrated water depends on porosity and hydraulic conductivity as indicated before. In other words, $\partial h/\partial t$ is approximately equal to infiltration rate.

Until now, Z_t is the closest we have reached to the infiltration rate. Recalling that Z_t is the total addition to the water-resource, it is obvious that the potentiometric head does not increase with the size Z_t. Z_t needs to be rewritten according to the potentiometric surface. Since the potentiometric surface is totally dependent to amount of recharged water it will increase as much as $Z_t/\varphi dxdy$ in each period of time, where it shall be written as a maximum intrinsic per head unit following is true $Z_t/\varphi dxdydz$. If $Z_t/\varphi dxdydz$ is defined as done in following equation

$$z_t = \frac{Z_t}{\varphi dxdydz}$$

then $\partial h/\partial t$, infiltration rate, can at this point be interpreted mathematically as

$$(E2.3) \qquad \frac{\partial h}{\partial t} = z_t$$

Furthermore, z_t can be defined as the maximum intrinsic per volume unit ("per capita") growth rate of the mass of water ("population") (Wilen, 1985, p. 72), where *m* is equal to $\varphi dxdydz$. Therefore z_t can be defined in following matter,

$$z_t = \frac{Z_t}{m_t}$$

It would be very convenient for the simulations later on in the paper to say something about the exact value of z_t but it's a very complicated matter. According to Wilen (1985, p. 81) the following formula could be used as an indicator for it,

$$z_t = \frac{1}{t_c}$$

Where

t_c = System's characteristic return time

System's characteristics return time is the time it takes for a system to return to equilibrium following a disturbance. If it takes a year then t_c is equal to 1. This property is very different between water systems and needs further examination. According to Wilen (1985, p. 87) this value should have the range of $0 \leq z_t \leq 2{,}692$ under natural condition. To be able to make simulation later on, z_t is assumed to be equal to 1. According to the equation above it means that it takes the water system one year to return to equilibrium following a disturbance.

According to equation (*E*2.3) the head will always be influenced by maximum z_t. One has to keep in mind that z_t is the maximum intrinsic per head unit, meaning that it is the maximum that precipitation can offer in period *t* when the evaporation and transpiration have been subtracted. There is one thing missing. It is the ground's varying property to absorb water with respect to its state of moist. As indicated before in chapter 2.4.1 and 2.4.2, it is at its maximum when the ground is completely dry and at its minimum when the ground is completely saturated.

<u>The infiltration is proportional to the aquifer size, *m*</u>. It is helpful to demonstrate how the infiltration rate changes with time, which e.g. is argued by Fetter (1980, pp. 87-89) and Otnes and Ræstad (1978, pp. 58-60). Infiltration rate falls with time. The ground has its best ability to receive the precipitation in the beginning: The drier the ground is the better the ground's ability to infiltrate the rain. The infiltration rate could be defined mathematically as:

$$(E2.4) \qquad \frac{\partial h}{\partial t} = z_t \left(1 - \left(\frac{h_t}{P_t} \right)^{\mu} \right)$$

Where

μ	=	Growth indicator in the aquifer: $0 < \mu < 10$.
P_t	=	*environmental carrying capacity*[19]: $P_t = dxdydz \div soil > 0$.
z_t	=	Intrinsic growth rate: $0 \leq z_t \leq dz \div (soil/3)$

The proportional growth rate, $\partial h/\partial t$, is approximately equal to z_t, when h_t is close to zero. z_t can therefore be seen as the intrinsic growth rate. The parenthesis is a number somewhere between and equal to zero and one. It is 1 when h_t is zero and zero when h_t is at its maximum, equal to P_t. The parenthesis ensures that the equation takes the influence from size of the aquifer on rejected water into account. The less water that is in the aquifer, the less becomes the volume of rejected water. The greater volume of water is in the aquifer, the greater is volume of rejected water.

The context between the volume of an aquifer and the head of an aquifer is known to be positive. An increase in mass of water in storage must lead to an increase in potentiometric surface. So potentiometric surface (head) and volume (aquifer) are concepts for the state of water in storage. The relationship between them is assumed to be like:

$$(E2.5) \qquad h_t = \psi m_t$$

Where

$$\psi^{20} \quad = \quad \frac{1}{dxdy} : 0 < \psi < 1$$

This assumption means that one unit of mass does not give one unit of head. One unit of mass gives less than one unit of head. This is because the head, h, is a one-dimensional unit while mass, m, is three-dimensional. Besides, one has to regard the natural shape of an aquifer. If the saturated zone increases this shape will lead to a greater area of soil covered by the potentiometric surface. This shape will reduce the

[19] Definition: An *environmental carrying capacity* is the total maximum of water which one specific aquifer can contain under natural condition.

[20] Eduction: From figure 2.4 it is proper to claim that $h=dz$ and $m=dzdxdy$. Then $m=hdxdy$ and following must be true $h=m/dxdy$.

ψ-value further. However, since the aquifer has the shape of the control volume this will not be the problem in this paper.

Figure 2.5

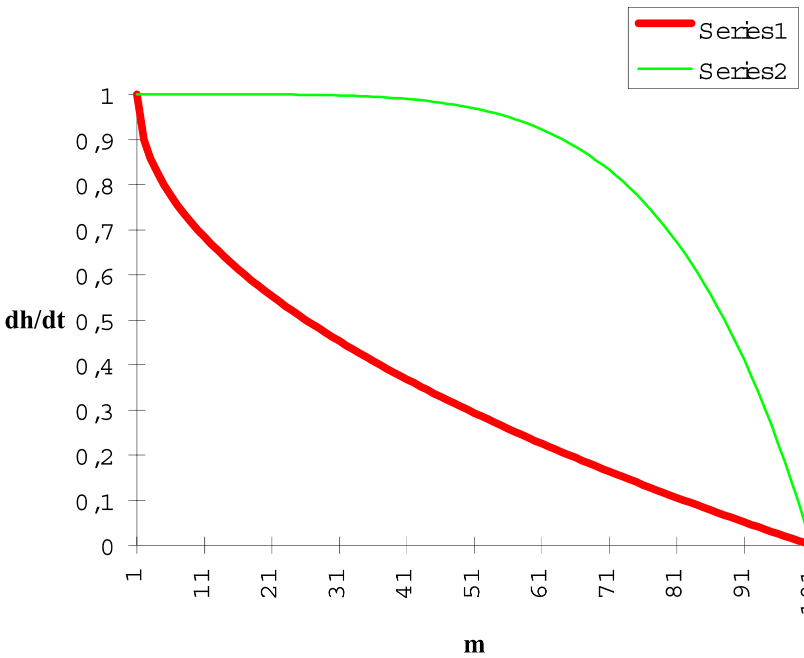

Figure 2.5 is a *phase diagram*[21] which shows how the head changes with time, $\partial h/\partial t$, when the water-mass, *m*, changes. There are two series. The difference between them is flow. Series 1 stands for faster flow than series 2.

However this inconvenience could be avoided by defining the environmental carrying capacity in terms of mass of water (m^3) instead of water level (potentiometric surface, *h*). According to this, the ψ-value has to be defined as equal to 1. This is what we do in this paper, but to keep the model as general as possible, ψ remains in the equation. Now, imbedding equation (*E*2.5) into equation (*E*2.4),

$$(E2.6) \qquad \frac{\partial h}{\partial t} = z_t\left(1-\left(\frac{\psi m_t}{P_t}\right)^{\mu}\right)$$

[21] Explanation: The *phase diagram* is a tool, which is suitable for presentation in dynamic analysis. Further reading about *phase diagram* is given by Chiang (1984, p. 493-496).

The value of μ influences ∂h/∂t. With a higher μ the head changes faster with time (greater ∂h/∂t). The faster the head changes with time the slower is the flow in the aquifer: The slower the flow responds to the pressure in the aquifer, the higher the μ ("Series 2" in figure 2.5). Thus, the faster the flow responds to the pressure in the aquifer the lower the μ ("Series 1" in figure 2.5). In "Series 1" μ is equal to 0,5. In "Series 2" μ is equal to 5.

With this argumentation the "flow of groundwater" represents the aquifer's outflow or discharge (q_x in figure 2.4 is only withdrawing water) and the infiltration represents the inflow or recharge (q_z in figure 2.4 is only adding water). This argumentation is assumed to be appropriate in this paper.

Now, equation (*E*2.6) is substituted for ∂h/∂t in equation (*E*2.2),

$$(E2.7) \qquad \frac{\partial m}{\partial t} = m_t z_t \left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right)$$

In the context of this paper the environmental carrying capacity is unique in each of the states of dynamic equilibrium. For each rate of precipitation there exists a unique state of dynamic equilibrium. The environmental carrying capacity, P_t, is therefore a function of the water-quantity that becomes ground water, z_t, and it is assumed to be mathematically defined by the following equation:

$$(E2.8) \qquad P_t = P(z_t) = B \ln \mathrm{z}_\mathrm{t}$$

Where

B = A constant: $B > 1$.

This is true for P_t when the environmental carrying capacity expands only vertically. This is assumed to be true in the context of this paper. Equation (*E*2.8) substituted for P_t in equation (*E*2.7):

$$(E2.9) \qquad \frac{\partial m}{\partial t} = z_t m_t \left(1 - \left(\frac{\psi m_t}{B \ln z_t}\right)^{\mu}\right)$$

The equation (*E*2.9) is the growth model of the aquifer that is shown in figure 2.7. It is

Pearl-Verhulst logistic model (Clark, 1976, pp. 10-16; Wilen, 1985, 71-77).

Figure 2.6

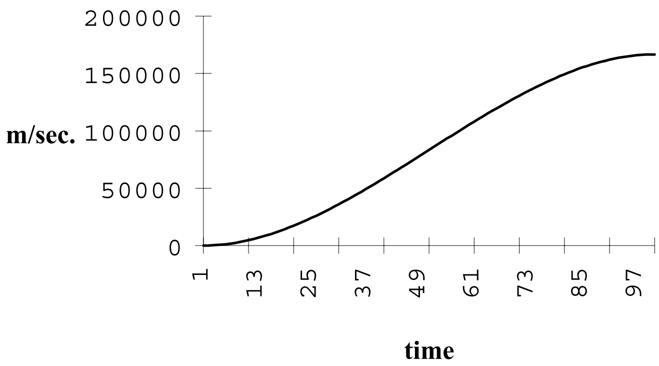

Figure 2.6 shows how the flow responds when the precipitation rate increases with respect to time.

The reason why it is suitable to use it on water, even though the author has not seen it before is that Wilen (1985, 71) says that its proper to use it for renewable natural resources where its “population” grows as a S-curve over time (see figure 2.6). According to Burchart and Jörgensen (1976, p.141) it is reasonable to assume the growth of water resources as an S-curve over time. Wilen (1985, 89) continues by pointing out that such a model can be used on renewable natural resources phenomenon that is stock or flow related. This has already been shown as central concepts of the growth of the water resource. One should also have in mind that even though water does not “give birth to” water the size of a water resource influences its marginal growth in similar way as the size of a fish stock does.

The growth of the aquifer can be described in the following matters. When the aquifer (m_t) is very small, the precipitation leads to a high infiltration rate (in m^3/second), or the inflow rate. Initially, the marginal increase of the vertical part of the aquifer (control volume) is faster than the horizontal part (figure 2.4). A vertical accumulation in the aquifer increases pressure that decreases the infiltration rate (inflow rate) and the horizontal growth rate of the aquifer. After a while, the marginal growth of the horizontal part of the flow reaches the vertical part and then exceeds it. Finally, when the aquifer reaches its environmental carrying capacity then the *vertical growth*[22] of the flow is equal to the *horizontal growth*[23] (meaning that the inflow is equal to the outflow). Hence, the accumulation of water in the aquifer stops and then the growth of the "dynamic aquifer" ($dm/dt = 0$) stops also.

Figure 2.7

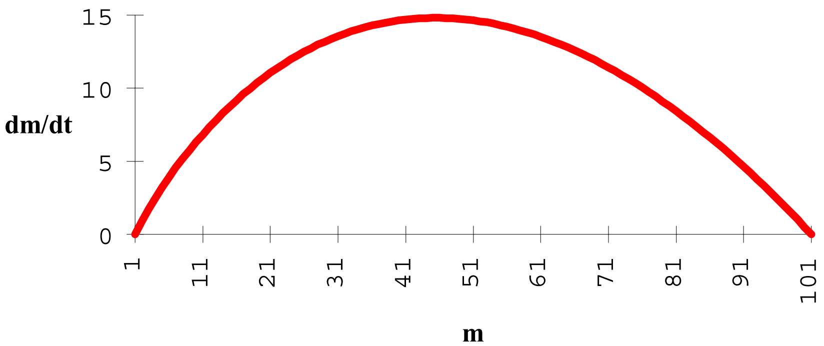

Figure 2.7 is a phase diagram, which shows the graphical extension of the growth model. It shows how the water-mass changes per time unit when the mass changes.

Another way of explaining figure 2.6: Initially, the aquifer (m_t) is very small and the precipitation leads to high infiltration rate, in m3/second. When the aquifer (m_t) increases, the pressure increases also and causes the flow (exponentially). For $m_t < m_{MSY}$, proportionally more of the infiltration goes into building up the aquifer than compensating the flow ($dq_Z/dm > dq_X/dm$ in figure 2.4).

$$m_{MSY} = \text{Maximum sustained yield}^{24}$$

For $m_t = m_{MSY}$ ($m_{MSY} \approx 50$ in figure 2.7), equal proportion of the infiltration goes to building up the aquifer and compensating the flow ($dq_Z/dm = dq_X/dm$ in figure 2.4). For $m_t > m_{MSY}$, proportionally less of the infiltration goes to building up the aquifer than compensate the flow ($dq_Z/dm < dq_X/dm$ in figure 2.4). When the mass reaches its

[22] Explanation: *Vertical growth* means that the aquifer expands upwards, opposite direction to q_z in figure 2.3.
[23] Explanation: *Horizontal growth* means that the outflow in the aquifer increases, in the same direction as q_x in figure 2.3.
[24] Definition: *Maximum sustained yield* is the aquifer's water stock (level) at which the productivity of the water resource is maximised (Clark, 1976, p. 13).

environmental carrying capacity (in $m_t = P_t$) the whole infiltration does nothing else than compensate the flow. Thus, the growth of the aquifer and the effective aquifer stops ($q_z = q_x$ in figure 2.4).

Attention should be paid to the difference between the two series in figure 2.8, found in an Excel 4.0 simulation of the growth function. The point maximum has changed because of the μ value. In "Series 1" the value of μ is 0,5. In "Series 2" the value of μ is 5. When the value of μ is large it means that the flow has a slow response to the pressure in the aquifer. The slower the flow that responds to the pressure in the aquifer the larger the m_{MSY} and the dm/dt, given equal environmental carrying capacity, P_t.

Figure 2.8

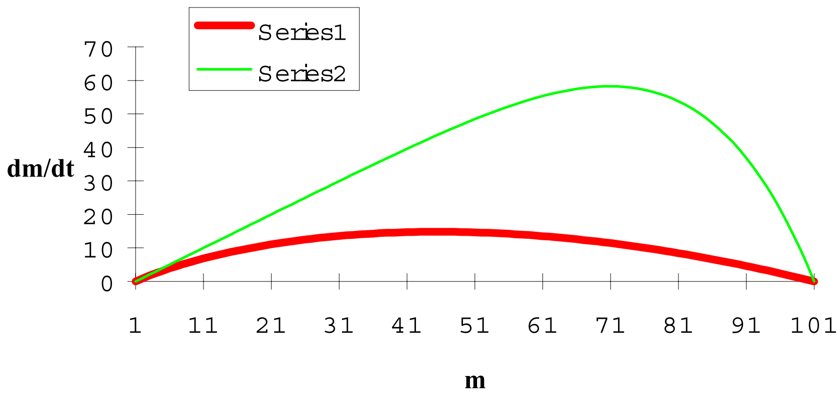

Figure 2.8 is a phase diagram that shows how the water-mass changes per time unit when the mass changes. There are two series. The difference between them is the velocity of the flow. Series 1 stands for a faster flow than series 2.

Notice the difference between the two series in figure 2.9 found in Excel 4.0 simulation of the growth function. The maximum point has changed because of the value of z_t, which is the arriving volume of water to the ground surface. It is smaller in "Series 1" than in "Series 2". The greater precipitation is the larger are m_{MSY}, dm/dt and P_t. The precipitation will influence m_{MSY}, dm/dt and P_t positively but marginally decreasing.

Figure 2.9

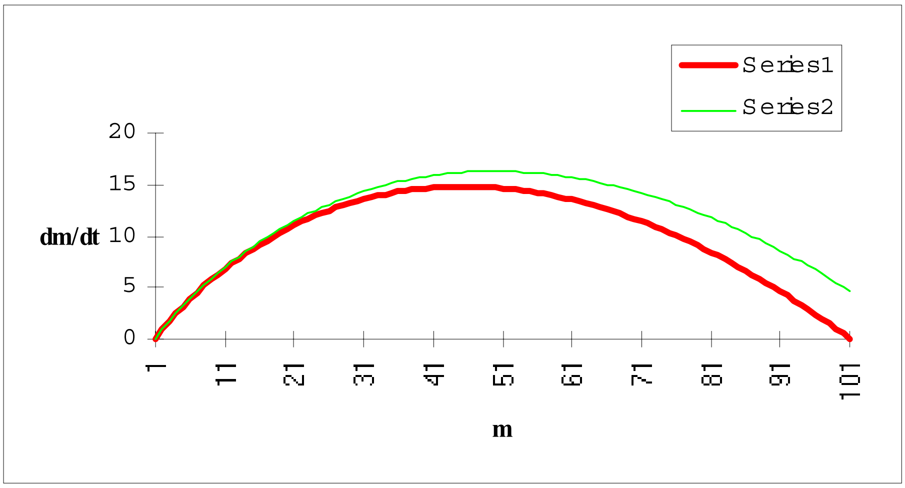

Figure 2.9 is a phase diagram that shows how the water-mass changes per time unit when the mass changes. There are two series. The difference between them is the precipitation rate. Series 1 stands for a lower precipitation rate than series 2.

2.4.4 Dynamic constraint for ground water system.

The well and the pumping cone

To reach water from an aquifer it is necessary to make a well. When the pumping begins the water is drawn from storage around the well and from the vertical flow. As the *pumping cone*[25] in the well grows, an increasingly larger portion of the aquifer will release water from the storage (see figure 2.10).

[25] Definition: A *pumping cone* is a cone shaped volume of not saturated soil with well in its centre. It is a result from extraction of water from an aquifer.

Figure 2.10

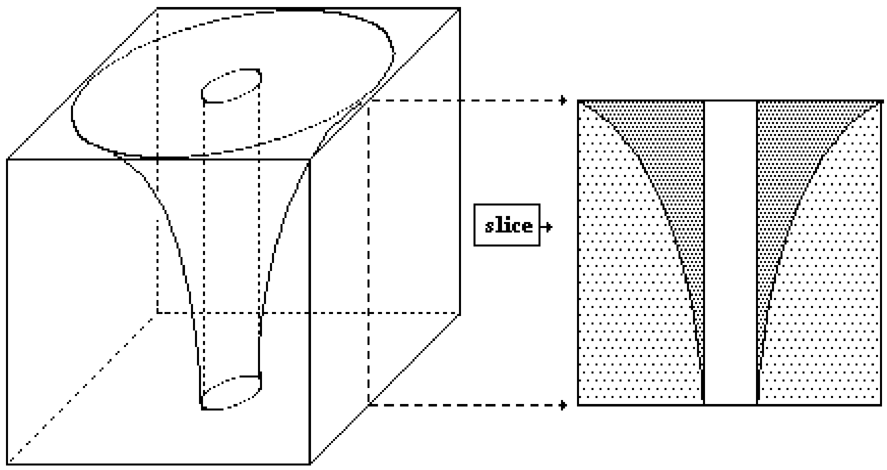

Figure 2.10 shows the control volume. A pumping well is in the centre of the control volume. The pumping cone is around it. The pumping cone can be seen even clearer in its cross section (called "slice" in the figure). The darkest area is the pumping cone, with unsaturated soil. The white column in the centre is the well. Finally, the grey areas nearest to both edges, are saturated areas.

The following equation describes the pumping from a well of a confined unit and how it creates a pumping cone around the well. It is called the drawdown equation (Fetter, 1980, p. 170), evaluated by C.E.Jacob and is valid for small values of *r* and large values of *t*.

$$\eta_0 - \eta = \frac{2{,}3Q}{4\pi T} \log_{10} \frac{2{,}25Tt}{Sr^2}$$

Where

Q	=	The constant pumping rate
η	=	The local hydraulic head at time t since pumping began
η_0	=	The local hydraulic head before the start of pumping
r	=	The radial distance from the pumping well to the observation well

Chiang (1967, p. 286) stresses the following:

$$\log_{10} N = (\log_{10} e)(\log_e N)$$

The exact number substitutes for the expression

$$\log_{10} N = 0{,}4343 \log_e N$$

This context substitutes the relevant term of the drawdown equation.

$$\eta_0 - \eta = (0{,}4343)\frac{2{,}3Q}{4\pi T}\ \ln\frac{2{,}25Tt}{Sr^2}$$

Multiplying 0,4343 and 2,3 together, the drawdown equation becomes equation (*E*2.10)

$$(E2.10) \qquad \eta_0 - \eta = \frac{0{,}99889Q}{4\pi T}\ \ln\frac{2{,}25Tt}{Sr^2}$$

The drawdown equation is the function of a "radial distance", *r*; which is the distance from the pumping well to the location of the calculated drawdown. Storativity, *S*, which is relatively small (0,005 or less) in a confined aquifer change with water level. The pumping does therefore influence a relatively large area. The constant pumping rate, *Q*, is the amount of water taken from the resource during a specific time, *t*. Aquifer transmissivity, *T*, describes the rate of the recharge to the drawdown and the well itself. Chapter 4.2.4 contains a discussion on how some of the most significant variable influences the drawdown, (η_0-η).

The pumping from the well will not influence the natural discharge until it reaches the discharge area. Then the potentiometric gradient toward the discharge area is lowered and the amount of natural discharge is proportionally reduced. When the pumping cone reaches the recharge area, it may lead to additional recharge of water that was previously rejected.

As mentioned before, the pumping cone will continue to grow until it has sufficiently reduced the natural discharge or increased recharge to balance the volume of water removed by pumping. At that moment, new condition of "dynamic equilibrium" is reached and the pumping cone remains in its unchanged shape:

> How does the pumping cone change with respect to time? The first partial derivative gives (*E*2.11):

$$(E2.11) \qquad \frac{\partial(\eta_0 - \eta)}{\partial t} = \frac{0{,}99889Q}{4\pi T^2}\left(\frac{1}{t}\right)$$

The expression shows that when the well is withdrawn by Q (as constant). It will have greatest influence on the formation of the cone in the first period and "almost" nothing in the later periods. The cone will keep its unchanged shape in a state of dynamic equilibrium by the constant pumping rate Q.

If the sum of the remaining natural discharge and the pumped water exceed the available recharge, the pumping cone will not stabilise and water levels will continue to fall ($m_t < m_{MSY}$). If it continues long enough, the aquifer could be seriously damaged. This had happened and some of the aquifers had never been usable again, because the resource collapsed. So, it is important to keep the aquifer in a dynamic harmony.

The dynamic constraint and the pumping cone

Recall that the equation of the growth of the aquifer (E2.9) is:

$$F(m_t;\bar{z}) = \frac{\partial m}{\partial t} = z_t m_t \left(1 - \left(\frac{\psi m_t}{B \ln z_t}\right)^{\mu}\right)$$

"It is inevitable that pumping from waterholes will cause <u>*some*</u> *local lowering of water table levels in order that groundwater will be induced to flow towards the wells."* (Brassington, 1988, p. 124). *"In any event, the pumping cone will continue to grow until it has sufficiently reduced natural discharge or increased recharge to balance the volume of water removed by pumping."* (Fetter, 1980, p. 445). From these expressions it is obvious that no matter how small the volume of pumped water is, it affects the whole aquifer (potentiometric surface) in <u>some</u> degree.

$$(E2.12) \qquad \overset{\circ}{m}_t = F(m_t;\bar{z}) - x_t$$

The influence from the pumped volume of water does not spread evenly over the potentiometric's surface. That is obvious from the formation of the pumping cone. The surface in the location nearest the waterhole will fall far below the potentiometric surface and then increase logarithmically from there. That is why every volume of water taken from the aquifer will not affect the static aquifer (potentiometric surface) relatively. The greater influence near the waterhole compensates the smaller influence further away. If it compensates completely, then ω = 1, otherwise 0 < ω < 1.

$$\overset{\circ}{m}_t = z_t m_t \left(1 - \left(\frac{\psi m_t}{B \ln z_t} \right)^{\mu} \right) - \omega x_t$$

Where

$\omega \quad = \quad \omega(\pi r^2, \text{number of wells}): 0 < \omega \leq 1.$

If the aquifer is confined or semi-confined, where it is surely confined in the extraction area then the rain will not be infiltrated there. Therefore it is most appropriate to assume 0 < ω < 1, but if the aquifer is unconfined the rain will be infiltrated wherever present. It is then best to assume ω = 1, which happens to be the assumption of this paper.

Conclusion to problem 1: The following conclusion is drawn from the analyses above: The expression for the dynamic constraint in the case of groundwater system turns out to be:

$$(E2.13) \quad \overset{\circ}{m}_t = z_t m_t \left(1 - \left(\frac{\psi m_t}{B \ln z_t} \right)^{\mu} \right) - x_t$$

Where

$\overset{\circ}{m}_t = \frac{\partial m}{\partial t} =$ How the water volume in the aquifer changes with respect to time.

<u>The marginal cases:</u> By keeping the water level in the aquifer close to being empty the pumping cone becomes large but the flow low. Then each unit of extracted water mostly comes from vertical growth and almost nothing from the horizontal growth. It means that each unit contains relatively much from the rain falling directly into the well but nothing from the rain falling far from the well. One is reaching relatively little of the rainfall. By keeping the water level in the aquifer close to full the pumping cone becomes small but the flow high. Then each unit of water extracted must mostly come from the horizontal growth but almost nothing from the vertical growth. It means that

each unit contains relatively much from the rain falling directly into the well but nothing from the rain falling far from the well, it reaches relatively little of the rainfall.

There is another constraint that has to be taken into consideration. This is called the quality constraint. It will be described in chapter 3.2.2. In the context of this paper it has been taken care of in the cost function, c_m, which is discussed in detail later.

2.5 Reservoir system

The following description of a reservoir system is a theoretical description of one that is a source of water supply in Bergen, the capital of west-Norway, which has a population of about 215.000 inhabitants.

2.5.1 Growth model for reservoir system.

A reservoir system is like a well in the ground water system, a man made "mechanism", which makes it possible to extract huge quantity of water at a time from the hydrologic cycle. These two systems have many things in common. They both have a tight relationship with the local rainfall zone. Therefore, it is appropriate to dismiss other sources such as other rainfall zones for the growth of water, as in the case of the ground water system above. The quantity of arriving water depends on the precipitation, and the size of the relevant rainfall zone. The rainfall zone is very stable from time to time and is assumed to be constant. As discussed before, the rainfall varies between months. Each month has its known mean value and probability for rainfall based on historical data. A fraction of the rainfall is caught by plants and transpired. Some of it evaporates or even falls in pathways that pass the water-magazine. Thus only a fraction of the whole rainfall in the relevant rainfall zone runs to the dam. So the equation (*E*2.1) in chapter 2.2 is highly relevant.

Water that runs to a dam usually comes from both overland flow and ground water. When the water has entered the water magazine it evaporates, transpires or flows from the magazine either through the sediments or through the wall. The difference is supplied to the *demanders*[26].

The water-reservoir has different ability to accumulate water. Imagine a dam is completely empty and suddenly it begins to rain, hard and constantly. Initially, the dam has its best ability to keep the entering water. This ability will contract with larger water-mass and its growth rate will decrease. Water will cover larger area in the dam and therefore find more and more ways through the sediments. Then the pressure from the water-mass will grow and boost the velocity of the water flowing from the dam through the sediments. The water-mass keeps on losing the net rate of growth (accumulation) because rate of outflow grows. If Z_t is large enough, the water level

[26] Definition: *Demanders* are only residences and companies that use water in order to maximise profit or utility. Wild animals and plants are not defined as demanders.

reaches the lowest top of the dam wall, that is the highest level of its environmental carrying capacity and water passes it and its growth stops. It is obvious that the growth of the water-mass in a reservoir system is proportional to its size. From the discussion above it is quite appropriate to assume that the reservoir system growth curve is represented by the following differential equation:

$$(E2.14) \qquad \frac{\partial s}{\partial t} = s\frac{\partial h}{\partial t}$$

Where

s = volume of water in the water magazine.

The size of the evaporation from the surface of the dam is uncertain since it depends on the solar radiation and the size of the surface where weather is thoroughly unpredictable. A larger surface, m^2, leads to a larger volume of evaporated water per time, where the surface increases with higher water level in the magazine. Hence, the greater the water level is in the dam, the greater is the evaporation.

The transpiration depends on the number of plants, trees particularly, and their need for water, which varies with for example weather, the solar radiation and so forth. The water level influences the transpiration: The higher the water level, the greater the number of roots reaching the water in the magazine and the greater is the transpiration.

The flow of water from the water magazines through the sediments depends on its water volume. The larger the water volume, the greater the pressure in the system and the greater the velocity of the waterflow from the magazine that gets through the sediments.

Finally, there is the environmental carrying capacity. There are many levels that are determined by the precipitation rate and the lowest level of the magazine wall. When the precipitation rate is very low, it cannot create water accumulation that fills the water magazine. The system then reaches a temporary environmental carrying capacity, $P_{temporary}$, as in the groundwater system) where the precipitation only compensates the evaporation, the transpiration and the flow through the sediments. With a sufficient precipitation rate the water-mass, s, in the magazine reaches the highest level of the environmental carrying capacity, P_{max}, which is known to be constant. The lowest wall of the dam will determine this constant. Each reservoir system has its unique P_{max}, where $s \leq P_{max}$.

The accumulation in the reservoir system is assumed only to have its origin in the Z_t. Then $\partial h/\partial t$ shows how water that runs to the dam changes its head with time. The proportion of the arriving water that creates accumulation of the water-mass in the water magazine is proportional to its size, h. The assumption about relationship between the water-mass and the head, demonstrated in equation (E2.4), is also assumed to hold. Thus, equation (E2.15) is a suitable equation for $\partial h / \partial t$ when the arguments are collected.

$$(E2.15) \qquad \frac{\partial h}{\partial t} = z_t \left(1 - \left(\frac{\psi s_t}{P(z_t)} \right)^{\sigma} \right)$$

The proportional growth rate, $\partial h/\partial t$, is approximately equal to z_t, for s_t close to zero. If the reservoir system gets completely empty, the biological mass will not collapse. This is why the system never shows a negative growth for a small s_t. Equation (E2.16) occurs when equation (E2.15) is substituted for $\partial h/\partial t$ in equation (E2.14),

$$(E2.16) \qquad F(s_t : \bar{z}) = \frac{\partial s}{\partial t} = s_t z_t \left(1 - \left(\frac{\psi s_t}{B \ln z_t} \right)^{\sigma} \right)$$

Where

σ	=	Growth indicator in the dam: $0 < \sigma < +\infty$.
z_t	=	Rate of arriving water: intrinsic growth rate.

It is likely that the impedance in the reservoir system is greater than the one in the groundwater system, because of the tight dam wall and the specially chosen location. However, where the groundwater system is naturally chosen and constructed, the impedance can be of various types. If these speculations are true, which seems to be very suitable assumptions because the sediments are generally tighter in West Norway than in Reykjavik, then the following will be true;

$$(C.3) \quad \sigma > \mu$$

However this is only true when the reservoir system has $P_t < P_{max}$.

2.5.2 Dynamic constraint for reservoir system.

It is obvious that no matter how small the volume of pumped water is, it affects the whole aquifer (potentiometric surface) in some degree. The influence from the pumped volume of water does spread evenly over the dam surface.

$$\overset{\circ}{s}_t = F(s_t; \bar{z}) - x_t$$

Conclusion to problem 2: From the analyses in chapter 2.5 this conclusion follows: The expression for the dynamic constraint of in the case of reservoir system turns out to be:

$$(E2.17) \qquad \overset{\circ}{s}_t = z_t s_t \left(1 - \left(\frac{\psi s_t}{B \ln z_t} \right)^{\sigma} \right) - x_t$$

The marginal cases: Every amount of entering water goes either to compensating the outflow or building up the water-resource. When the water level in the aquifer is close to being empty the proportion of the entering water that is building up the water-resource tends to infinity. The other, which is compensating the outflow, tends to zero ((dq_Z/dm) > (dq_X/dm) in figure 2.4). When the water level is close to being full, the proportion of the entering water that is building up the water-resource tends to fall to zero. The other, which is compensating the flow, tends to rise to infinity ((dq_Z/dm) < (dq_X/dm) in figure 2.4).

Another constraint that has to be taken into consideration is called the quality constraint. It will be described in chapter 3.3.1. In the context of this paper it has been taken care of in the cost function, c_s, which will be discussed later.

Chapter 3

3 The model

3.1 Introduction.

In chapter 2 two specific dynamic constraints of two types of water-resources were argued as the most appropriate descriptions. This is illustrated by taking two separate societies as examples. Each society has only one type of water-resource. They are the same as described in chapter 2. The purpose of this chapter is to develop the results of chapter 2 and use them in a calculation of the maximum social welfare. The calculation is based on a known method called optimal control theory. The mathematical expression is found for the maximum social welfare and it used to reach the objective of this chapter that is to describe the optimal solution for each society. *The question asked is, does the system have an optimal solution?* In order to enlarge the understanding of the answer the factors that influence the optimal solution is emphasised and how they influence it. The second objective of the chapter is to compare the groundwater- and the reservoir system in Reykjavik and Bergen where their water supply is totally dependent on one of these systems. This will be supported graphically by a two-variable-phase-diagram. These two objectives can be reached by solving three following problems:

Problem 3: *What is the optimal solution for the society which has a groundwater system?*
Problem 4: *What is the optimal solution for the society which has a reservoir system?*
Problem 5: *What is the difference between the optimal solutions for the groundwater system and the reservoir system?*

3.2 Economic view of the ground water system

3.2.1 Social benefit

Society is a collection of different individuals. Water is necessary for every individual, since everyone needs a minimum quantity of it each day. Therefore every individual is a water consumer. Where there is only one water-supplier water is a *homogeneous good*[27]. Since water satisfies some of basic needs such as drinking and washing, it is appropriate to assume in this paper that all consumers have the same consumption pattern of water. Accordingly, the assumption of a *representative consumer*[28] is proper.

Figure 3.1

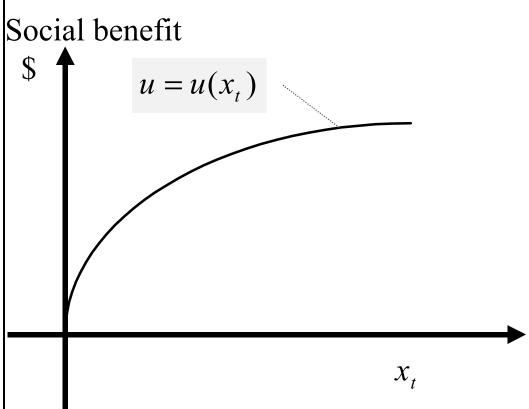

Figure 3.1 shows how the social benefit as a function of the supplied quantity of water. If the supplied quantity of water increases the social benefit increases.

The marginal *social benefit*[29] of water consumption can be defined mathematically as,

$$u = u(x_t)$$

Where

x_t = Quantity of water supplied to the consumer in period *t*.

u = consumers utility of water consumption.

The utility is assumed to be decreasing like have been done in similar studies (Shu and Griffin, 1992, p. 197; Tsur and Zemel, 1995, p. 151). This is demonstrated graphically in figure 3.1. A proper mathematical notation is

$$u'(x_t) > 0 \qquad u''(x_t) < 0$$

This seems to be a reasonable assumption since the first unit of water contributes the

[27] Definition: *Homogeneous good* is "*in any given market, the products of all firms are identical*" (Eaton and Eaton, 1988, p. 259; Brown, 1995, G-10).

[28] Definition: *Representative consumer* is *"the preference and income of an individual randomly selected from the group will be characteristic of each person in that group"* (Eaton and Eaton, 1988, p. 103-104).

[29] Definition: *Social benefits* include all those effects of a policy change that increase social welfare (Hardwick, Khan and Langmead, 1990, p. 134).

consumer a maximum utility, i.e. a matter of life or death and the last unit of water can give him negative utility, i.e. when one is drowning in water. Negative utility is however not dealt with in this paper. Expressing these mathematical terms in words, the utility function $u(x_t)$ is assumed to be increasing and strictly concave as expressed above and in figure 3.1. As pointed out in chapter 3.3, this is a proper assumption here.

3.2.2 Cost of supplying from the ground water system

Whenever the concept cost is used without any explanation in this paper it is a reference to the concept *social cost*[30]. The electricity allocates to the pumps is one of the major parts of the *variable cost*[31], which is somewhere close to 2 IKR per 1000 litre of water. That is somewhere among 14-16% of the total cost for to the supplier. Maintenance (e.g. in the conveyance system) is another significant part of the variable cost (Becker and Easter, 1992, p. 347). The cost can be defined as,

Figure 3.2

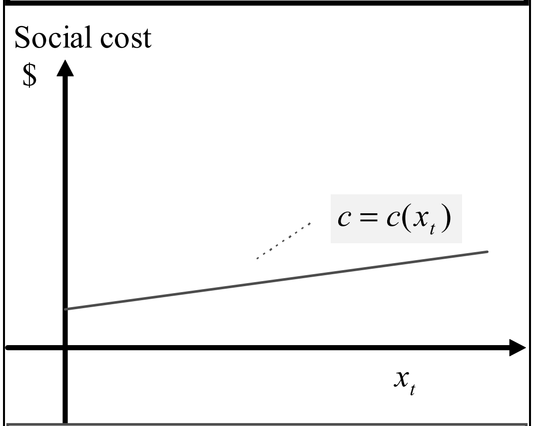

Figure 3.2 shows how the cost function is as a function of the supplied quantity of water. If the supplied quantity of water increases the cost increases.

(C.4) $$c = c(x_t)$$

Where

c = Social cost of water.

When the supplier pumps, it is usually at a constant rate because the first location of the water after it leaves the aquifer (dam) is a collection tank. This collection tank is placed higher than the households are. Thus the water can flow from there by the law of gravity. According to this it is appropriate to assume *marginal cost*[32] to be positive and constant (see figure 3.2), which is also stressed by Tsur and Zemel (1995, p. 151) and Neher (1990, p. 261):

---

[30] Definition: *Social costs* include all those effects of a policy change that reduce social welfare (Hardwick, Khan and Langmead, 1990, p. 134).

[31] Definition: *Variable cost* is certain costs in the short run that depend on the level of output or consumption individuals have chosen (Case and Fair, 1989, p. 196; Brown, 1995, G-26).

[32] Definition: *Marginal cost* is the rate of increase of variable cost as output increases; that is, it is approximately $\Delta c / \Delta m$. This is relevant in the short run. (Eaton and Eaton, 1988, p.202; Brown, 1995, G-14).

$$(C.5) \quad \frac{\partial c(x_t)}{\partial x_t} > 0 \quad ; \quad \frac{\partial c^2(x_t)}{\partial x_t^2} = 0$$

The supplier in Reykjavik can supply water directly from the well without a treatment or any kind for purification. Under natural condition the water in the well is pure (Carpenter, 1977, p. 431; Orkustofnun, 1989, p. 4) and has never been threatened by acid rain (Orkustofnun, 1989, p. 4). However, it could be polluted by other external factors, for instance household waste disposal or pollution caused by traffic. The supplier controls the area of the *inner aquifer*[33] by keeping its surface clean and building a fence around it. In 1984 all wells could be defined as closed wells. For this reason the probability for external pollution was further reduced.

Figure 3.3

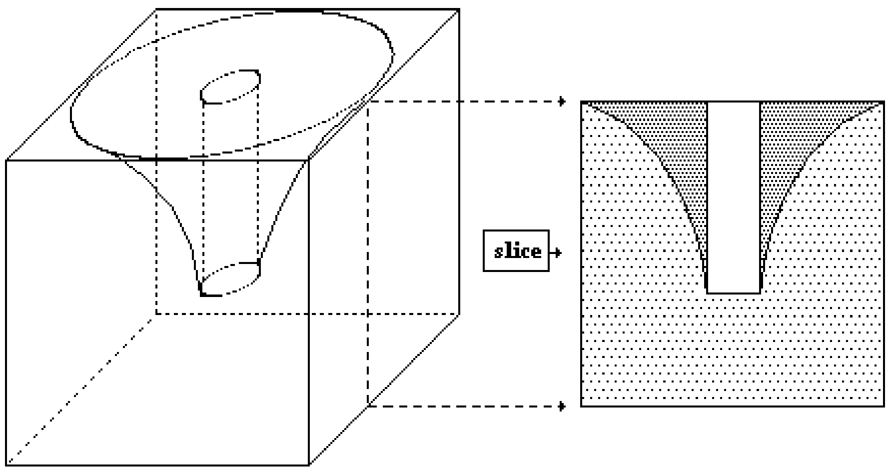

Figure 3.3 shows the control volume. A pumping well is in the centre of the control volume. The pumping cone is around it. The pumping cone can be seen even clearer in a cross section (called "slice" in the figure). The darkest area is the pumping cone with the unsaturated soil. The white column in the centre is the well. Finally, the grey areas nearest to both edges, are saturated areas.

The supplier is also forced to control the water flow through the aquifer. The inner aquifer must be protected from a possible inflow from the uncontrolled aquifer. In this paper the inner aquifer is assumed to be equal to the control volume. The main flow direction is known to be in the x-direction in figure 2.4. In order to let the uncontrolled

[33] Definition: *Inner aquifer* is the part of the aquifer, which the supplier has protected from possible external pollution to guaranty water's quality (See figure 3.6).

aquifer pass the inner aquifer one has to keep the water level in the well above some measurable minimum, see equation (*E*2.10). This is the quality constraint. If the control volume is assumed to be the inner aquifer this means that the pumping cone is not allowed exceeding it (see figure 2.4, 2.10, 3.3 and/or 3.4).

The quality constraint is not constant. It depends on the size of the aquifer (or the potentiometric surface). The greater the size of the aquifer (or the higher potentiometric surface) the higher will this minimum be in meters above the ocean (see the difference in figure 3.3 and 3.4). This means that if the difference between the height from the water level in the waterhole and the potentiometric surface can be greatest when the aquifer is full. This is caused by the flow of the groundwater through the aquifer. The water is taken from the horizontal accumulation of the aquifer. Therefore each unit taken makes less influence on the pumping cone when the aquifer is full than when it's not full. *A higher potentiometric head reduces the pumping influence on the drawdown* (chapter 4.2.4). Accordingly, if the water volume is low in the aquifer then *ceteris paribus,* the extraction limits will decrease.

Figure 3.4

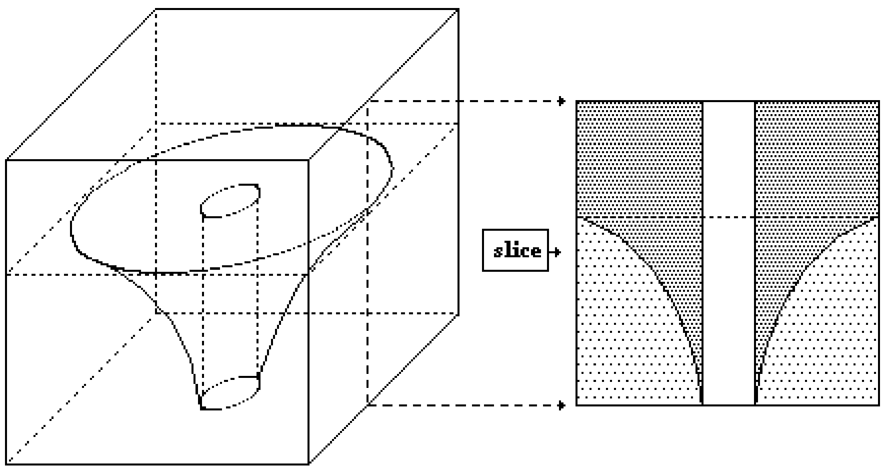

Figure 3.4 shows the control volume. A pumping well is in the centre of the control volume. The pumping cone is around it. The pumping cone can be seen even clearer in a cross section (called "slice" in the figure). The darkest area is the pumping cone with the unsaturated soil. The white column in the centre is the well. Finally, the grey areas nearest to both edges, are saturated areas.

The water holes are distributed over two sections, north and south. The average depth of the water holes in the south-section is greater than the ones in the north section, which means the cost of extraction from the south section would be higher. Hence, the cost function is dependent on the stock of water in the aquifer. This type of cost is related to *the stock effect*[34]. By taking the stock's influence to the social cost into consideration, context (*C*.4) changes to context (*C*.6).

$$(C.6) \quad c=c(x_t,m_t)$$

As discussed before the following expressions are assumed to be true descriptions of the cost of extracting water (see figure 5.1) which was also assumed by Tsur and Zemel (1995, p. 151) and Neher (1990, p. 261), in the equation below.

$$(C.7) \quad c(x_t,m_t) = c(m_t)x_t$$

Figure 3.5

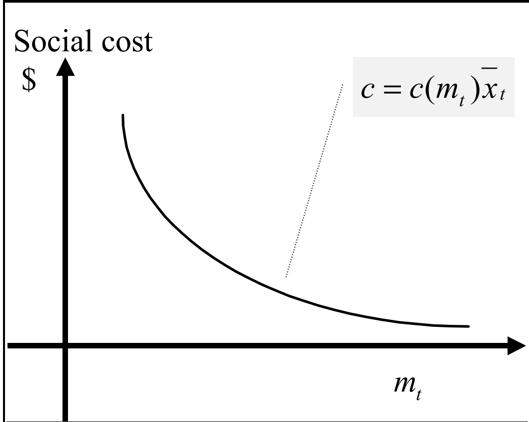

Figure 3.5 shows how the cost function is as a function of the total stock effect. If the volume of saturated zone increases the total stock increases and the total cost falls.

So, if the water volume in the aquifer falls, the well approach its quality constraint and the supplier is forced to extract proportionally more water from the south section than the north one. The following mathematical expressions describe this context,

$$\frac{\partial c(m_t)}{\partial m_t} < 0 \quad ; \quad \frac{\partial c^2(m_t)}{\partial m_t^2} \geq 0$$

The cost function, $c(x_t, m_t)$, is decreasing and convex in m_t as shown in figure 3.5. The suppliers gauge the quality constraint continually using a computer system. The pumping fraction from the sections changes automatically. The system extracts proportionally more water from the south section if the stock of water (aquifer) falls, meaning that the cost increases if the stock of water falls (see figure 3.5).

There are other cost related to the stock of water in the resource. This is loss of amenity, wildlife habitat and recreational possibilities (Pearce, 1993, p. 73). This cost does nothing else than support previous arguments regarding stock's influence to the

[34] Definition: *The stock effect* is the partial differentiate of the social cost with respect to stock (Neher, 1990, p. 314)

social cost.

Figure 3.6

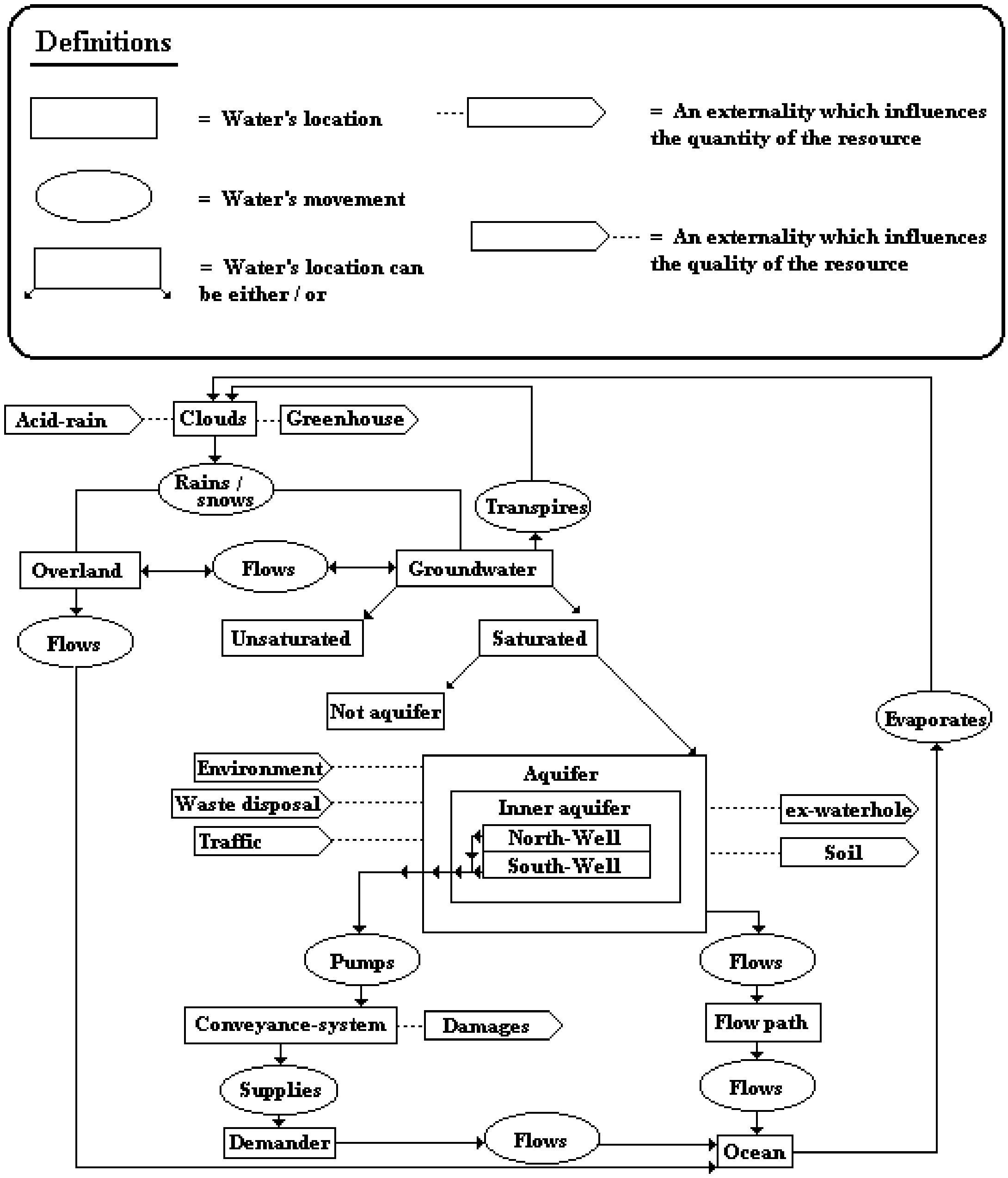

Figure 3.6 shows the context of the hydrologic cycle, given that a well has been placed into it.

Writing $(C.5)$ according to $(C.7)$ it becomes

$$(C.8) \qquad \frac{\partial c(x_t, m_t)}{\partial x_t} > 0 \qquad ; \qquad \frac{\partial c^2(x_t, m_t)}{\partial x_t^2} = 0$$

$c(x_t, m_t)$ increases in x_t as shown in figure 3.2.

Groundwater - and reservoir systems have many cost factors in common. That applies to most of the *fixed cost*[35]. Fixed cost is in two categories, "active fixed cost" and "non active fixed cost". The "active fixed cost" is the one that disappears when the supply is 0 but is constant when there is any positive supply. Salaries and small equipment are good examples. The "non active fixed cost" is constant in any state of supply in the *short run*[36]. Rent payments and depreciation are good examples. Even if this is not significant in the context of this paper it should be mentioned.

By collecting central facts and concepts of the ground water system together and add it into the context of the hydrologic cycle, figure 2.2 changes to figure 3.6.

3.2.3 Maximisation of the ground water system

Like Havenner and Tracy (1992, pp. 197-201), Neher (1990, p. 261) and /or Tsur and Zemel (1995, p. 151), the following model will be used:

$$\underset{x_t}{MAX}\ \Pi = \int_0^{\infty} \left[u(x_t) - c(m_t)x_t\right] e^{-\rho t}\, dt$$

Where

Π = Social welfare
ρ = Social discount rate: $0 < \rho < 1$.

This is the relevant object function. The objective is to choose the steady state for which the present value of sustained flow of "social welfare", $u(x_t) - c(m_t)x_t$, is the greatest (Neher, 1990, p. 44). Notice that the social welfare is discounted according to the continuous-discounting formula as Chiang (1984, p. 281) calls it.

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35 Definition: *Fixed cost* is costs which firms must bear in the short run regardless of their output. (Case and Fair, 1989, p. 196; Brown, 1995, G-9).
36 Definition: *Short run* is a time period within a month.

Figure 3.7

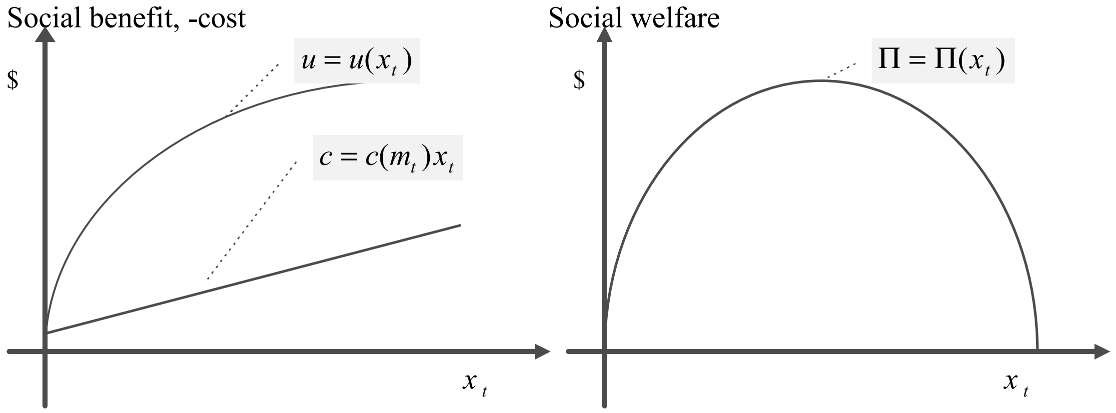

Figure 3.7 shows the social welfare graphically as a function of supplied quantity in the right panel. It shows the social benefit and the social cost together graphically as a function of supplied quantity in the left panel. This is relevant in the society which has a groundwater system.

Joining the discussions of chapter 3.2.1 and 3.2.2 (figure 3.1 and 3.2), figure 3.7 describes the object function graphically. This is a static description of it. The cost of extracting x_t at state m_t is $c(m_t)x_t$, and the benefit of consuming x_t is $u(x_t)$. The social welfare of consuming x_t at the state m_t is $u(x_t) - c(m_t)x_t$.

Recall the side term (the dynamic constraint) from equation (*E*2.13) that has already has been discussed for in chapter 2:

$$(E3.1) \qquad \overset{\circ}{m}_t = F(m_t; \bar{z}) - x_t$$

$$(E3.2) \qquad m(0) = m_0$$

Where

$$(E3.3) \qquad F(m_t; \bar{z}) = z_t m_t \left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right)$$

The *Hamiltonian*[37] function follows,

$$H = \left[u(x_t) - c(m_t)x_t\right]e^{-\rho t} + \lambda_t \left(z_t m_t \left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right) - x_t\right)$$

[37] Definition: *Hamiltonian* is interpreted as a performance indicator for the natural resource industry (Neher, 1990, p. 165).

where x_t is the *control variable*[38]*,* and m_t is the *state variable*[39] the *necessary condition*[40] of the Hamiltonian above, it follows that

$$(E3.4) \quad \text{(MP)} \quad \frac{\partial H}{\partial x_t} = \frac{\partial u(x_t)}{\partial x_t} e^{-\rho t} - c(m_t)e^{-\rho t} - \lambda_t = 0$$

$$(E3.5) \quad \text{(PB)} \quad \frac{\partial H}{\partial m_t} = -\overset{\circ}{\lambda}_t = -\frac{\partial c(m_t)}{\partial m_t} x_t e^{-\rho t} + \lambda_t \frac{\partial F(m_t;\bar{z})}{\partial m_t}$$

$$(E3.6) \quad \text{(DC)} \quad \overset{\circ}{m}_t = z_t m_t \left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right) - x_t$$

In the following we apply the notations;

$$\frac{\partial u(x_t)}{\partial x_t} - c(m_t) = V_t \quad ; \quad \frac{\partial c(m_t)}{\partial m_t} = c_m$$

$$\frac{\partial F(m_t;\bar{z})}{\partial m_t} = F_m \quad ; \quad \frac{\partial u(x_t)}{\partial x_t} = u_x$$

The second order condition or the *sufficient condition*[41] as it is often called is assumed to hold in this paper. Equation (E3.4) is the *maximum principle*[42] (MP). It ensures that the resource price is the difference between the marginal social benefits and the marginal social costs of using x_t. Equation (E3.5) is the *portfolio balance condition*[43] (PB). It reflects that the resource price must go up at the rate of interest, *Hotelling's rule*[44]*,* except if the resource contributes, marginally, to current performance as measured by the current Hamiltonian value. Since c_m is negative, society will be in

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[38] Definition: *Control variable* is the variable which can be controlled from moment to moment for the purpose of maximising, or minimising if that is the object (Neher, 1990, p. 163).

[39] Definition: *State variable* represents the momentary state of the world at any point in time, where the resource stock is immutable (Neher, 1990, p. 163).

[40] Definition: *Necessary condition: The* first order condition are necessary condition for the optimal solution (Neher, 1990, p. 165; Sydsæter, 1987, p. 353). It ensures that the chosen control variable gives the maximum value of social welfare with respect to every constraint (Sydsæter, 1987, p. 366)

[41] Definition: The *sufficient condition* ensures the sufficient shape (concavity or convexity) of the functions (the object function and the constraints) which is needed to confirm the solution of the first order condition (x^*, m^*) as the optimal solution (Sydsæter, 1987, pp. 353-360).

[42] Explanation: The *maximum principle* ensures that the chosen control variable gives the maximum value of accumulated social welfare with respect to every constraint (Neher, 1990, p. 166; Sydsæter, 1987, pp. 365-366). Further readings is given by Neher (1990, p. 166) and Sydsæter (1987, pp. 360 - 369).

[43] Explanation: The purpose of the *portfolio balance condition* is to ensure that the topic maximum problem is confronted by its opportunity cost. Further readings is given by Neher (1990, p.166).

[44] Definition: *Hotelling's rule:* Generally, the Hotellings rule is a portfolio balance condition for all assets that both can be freely shifted between portfolios in competitive markets. Besides it must be sterile in the sense that the holding of the stock of the asset per se yields no net benefits (Neher, 1990, p. 95; Sydsæter, 1987, p. 385).

portfolio balance for $\overset{\circ}{\lambda}_t / \lambda_t < \rho$, so the Hotelling's rule does not hold in this context. Equation (E3.6) is the *dynamic constraint*[45] (DC). It describes the motion of the water stock.

To find the growth models first partial differentiate (E3.4) with respect to stock ***(F_m)***, the functions are rewritten as below,

$$F(m_t;\bar{z}) = z_t m_t \left(1 - \left(\frac{\psi}{P_t}\right)^{\mu} m_t^{\mu}\right)$$

The partial differentiate from the formula above is:

$$\frac{\partial F(m_t;\bar{z})}{\partial m_t} = F_m = z_t\left(1 - \left(\frac{\psi}{P_t}\right)^{\mu} m_t^{\mu}\right) + \mu z_t m_t \left(-\frac{\psi}{P_t}\right)^{\mu} m_t^{\mu-1}$$

If m_t is cancelled in the second factor and joined the inner parenthesis of the first factor, it could also be written as,

$$\frac{\partial F(m_t;\bar{z})}{\partial m_t} = F_m = z_t\left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right) + \mu z_t \left(-\frac{\psi}{P_t}\right)^{\mu} m_t^{\mu}$$

The second factor may be rewritten by embedding m_t in the parenthesis. The expression changes to,

$$\frac{\partial F(m_t;\bar{z})}{\partial m_t} = F_m = z_t\left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right) + \mu z_t \left(-\frac{\psi m_t}{P_t}\right)^{\mu}$$

The common variable, z_t, is set outside outer parenthesis and the common variable of the two last factors, $\left(\frac{\psi m_t}{P_t}\right)^{\mu}$, is set outside the

[45] Explanation: The purpose of the *dynamic constraint* is to ensure that the topic maximum problem is confronted by the resources growth with respect to time. Neher (1990, pp.167-168) gives further readings.

inner parenthesis,

$$(E3.7) \qquad \frac{\partial F(m_t;\bar{z})}{\partial m_t} = F_m = z_t\left(1-(1+\mu)\left(\frac{\psi m_t}{P_t}\right)^{\mu}\right)$$

It will be confirmed that equation $(E3.7)$ can be negative or positive. It is positive for $m_t \in [0, m_{MSY})$ and negative for $m_t \in (m_{MSY}, P_t]$, where m_{MSY} is the stationary value.

Now finding the second partial differentiates of the growth model ($E3.4$) with respect to stock ***(F_{mm}).***

$$\frac{\partial^2 F(m_t;\bar{z})}{\partial m_t^2} = F_{mm} = z_t\left(-(1+\mu)\mu m_t^{\mu-1}\left(\frac{\psi}{P_t}\right)^{\mu}\right)$$

This can be rewritten.

$$(E3.8) \qquad \frac{\partial^2 F(m_t;\bar{z})}{\partial m_t^2} = F_{mm} = z_t\left(-(1+\mu)\frac{\mu}{m_t}\left(\frac{\psi m_t}{P_t}\right)^{\mu}\right) < 0$$

Equation $(E3.8)$ turns out to be always negative. It means that the stationary value is a maximum point. It confirms that equation $(E3.7)$ turns out to be both negative and positive. It is positive for $m_t \in [0, m_{MSY})$ and negative for $m_t \in (m_{MSY}, P_t]$, where m_{MSY} is the stationary value. This is also confirmed graphically in the simulation which lies behind figure 2.7, done in Excel 4.0.

To find the solution of the dynamic system, we use equation ($E3.4$) and equation ($E3.5$), to arrive at

$$(E3.9) \quad \frac{\partial H}{\partial m_t} = -\overset{\circ}{\lambda}_t = -c_m x_t e^{-\rho t} + V_t e^{-\rho t} F_m$$

Equation (E3.4) differentiated with respect to time:

$$(E3.10) \quad \frac{\partial H}{\partial x_t} = \overset{\circ}{V}_t e^{-\rho t} - \rho V_t e^{-\rho t} = \overset{\circ}{\lambda}_t$$

Where

$$\overset{\circ}{V}_t = \frac{\partial u^2(\mathrm{x}_t)}{\partial x_t^2}\frac{\partial x_t}{\partial t} - \frac{\partial c(m_t)}{\partial m_t}\frac{\partial m_t}{\partial t}$$

For simplicity we denote

$$\frac{\partial u^2(x_t)}{\partial x_t^2} = u_{xx}$$

$\frac{\partial x}{\partial t} = \overset{\circ}{x}_t =$ How the supply from the aquifer changes with respect to time.

Finally, the expression becomes like equation (E3.11) by using the definitions above,

$$(E3.11) \quad \overset{\circ}{V}_t = u_{xx}\,\overset{\circ}{x}_t - c_m\,\overset{\circ}{m}_t$$

Equation (E3.10) imbedded into equation (E3.9),

$$(E3.12) \quad \overset{\circ}{V}_t - \rho V_t = c_m x_t - V_t F_m$$

If ρV_t is added to both sides then the expression becomes,

$$\overset{\circ}{V}_t = c_m x_t - V_t F_m + \rho V_t$$

Equation (E3.11) substituted for $\overset{\circ}{V}_t$,

$$u_{xx}\,\overset{\circ}{x}_t - c_m\,\overset{\circ}{m}_t = c_m x_t - V_t F_m + \rho V_t$$

$c_m \overset{\circ}{m}_t$ added to both sides,

$$u_{xx} \overset{\circ}{x}_t = c_m (x_t + \overset{\circ}{m}_t) + V_t(\rho - F_m)$$

Equation (*E*3.6) substituted for $\overset{\circ}{m}_t$,

$$u_{xx} \overset{\circ}{x}_t = c_m \left(x_t + z_t m_t \left(1 - \left(\frac{\psi m_t}{P_t} \right)^{\mu} \right) - x_t \right) + V_t(\rho - F_m)$$

This reduces to

$$u_{xx} \overset{\circ}{x}_t = c_m \left(z_t m_t \left(1 - \left(\frac{\psi m_t}{P_t} \right)^{\mu} \right) \right) + V_t(\rho - F_m)$$

and can be rewritten as

$$(E3.15) \quad \overset{\circ}{x}_t u_{xx} = \left[c_m (F(m_t; \bar{z})) + V_t(\rho - F_m) \right]$$

Equation ($E3.15$) along with DC, equation (*E*3.6), is the synthesised dynamic system of necessary conditions for optimal management of the aquifer. If ($E3.15$) is equal to zero then the *steady state*[46] of equilibrium will be obtained. If ($E3.15$) > 0 then it is profitable for the society to take more out of the aquifer unit. If ($E3.15$) < 0 then it is profitable for the society to take less out of the aquifer unit.

In order to hold the steady state condition (*E*3.15) could be rewritten as:

$$(E3.15b) \quad F_m - \frac{c_m (F(m_t; \bar{z}))}{u_x - c(m_t)} = \rho$$

ρ is the discount rate. According to equation (*E*3.15) and (*E*3.15*b*) the discount rate, ρ, encourages increase in extraction from the resource. The exact value of the discount rate is assumed to be equal to and larger than zero and equal to and less than

[46] Definition: *Steady state* is a long run equilibrium where accumulation and technical change (dynamics) are included (Eatwell, Millgate, Newman, 1987, p.626). The same essence of understanding is given by Dornbush and Fischer (1978, p. 526) but it is coloured by a macroeconomic point of view. Bannock, Baxter, Davis (1972, p. 386) says following about steady state growth *"A feature of an economy in which all variables grow (or contract) at a constant rate… If these rates are maintained indefinitely, steady state growth exists".*

one, $0 \leq \rho \leq 1$. There are several views among economists about the interpretation of the discount rate. Three views will now be explained, shortly. Firstly it is the view that the *opportunity cost of capital*, that it ought to reflect a rate earning from comparable project. Secondly it is the view of *the cost borrowing money*. If the Government is in deficit then it has to lend money to finance projects. In this case the discount rate shall reflect the interest rate that the government has to pay its creditors. Finally it is *the social rate of time preference*. Then the discount rate shall reflect the community's preference of consumption today over consumption tomorrow (Dixon, Scura, Carpenter and Sherman, 1994, pp. 39-40; Pearce and Moran, 1994, pp. 21-28; Young and Haveman, 1985, pp. 488-489). If the consumer is impatient the discount rate is generally high. If he is not and let us say that he is immortal, the discount rate becomes equal to zero. From this point of view the discount rate becomes equal to zero if it is generally accepted that there should be equal rights between generations and there won't be an apocalypse. Further on, the discount rate will most likely increase on a warm and sunny day when the weather forecast for tomorrow is cloudy. Finally, the view of the discount rate in this paper will be the social rate of time preference. According to the discussion above there will be more pressure on extraction from the water resource in a generally impatient society.

$\boldsymbol{F_m}$ is the marginal growth of the aquifer. It has been discussed in details before. Under natural condition one specific rate of precipitation would create one specific environmental carrying capacity. When the pumping from the aquifer begins, the water volume in the aquifer falls below the environmental carrying capacity ($m_t < P_t$). While the marginal growth is negative (in $m_t > m_{MSY}$) the factor, F_m, encourages the increase of the pumping rate. While the marginal growth is positive (in $m_t < m_{MSY}$) the factor encourages reducing the pumping rate. So, if F_m was the only factor in $(E3.15)$, it would encourage to keep the aquifer where its vertical and horizontal growth rate is equal, meaning that the optimal solution would be $m_t^* = m_{MSY}$. It is somewhere between full and empty, which depends on the flow, that again depends on the factors like the hydraulic conductivity and the slope of the aquifer. The greater the flow in the aquifer at the P_t *ceteris paribus,* the further m_{MSY} from it. The smaller the flow in the aquifer at the P_t *ceteris paribus,* the nearer m_{MSY} from it.

$\boldsymbol{c_m}$ is the stock effect (marginal cost with respect to stock). If, as discussed before $c_m <$ 0 and $F(m_t;\bar{z}) > 0$, then the first factor encourages the decrease of pumping from the aquifer. As can be seen in figure 3.5, $|c_m|$ is largest in the first part of the defined domain of m_t $\left(m_t \in [0, m_{MSY}]\right)$ and becomes consistently smaller by consistently larger m_t. $F(m_t;\bar{z})$ however, is largest in the domain nearest to the maximum sustained yield $\left(m_t \approx m_{MSY}\right)$ and decreases consistently from it in both directions (see figure 2.7, 2.8 or 2.9). The stock effect, c_m, demands for some amount of water in the aquifer because of the increased costs of the water that was discussed in details before. One could say that the stock effect represents one aspect of water resources' vulnerability. The stock effect, $|c_m|$, can increase when it is progressively more expensive to extract water from the water-resource, given that the mass declines. This can happen when the electricity or the extraction from the alternative area becomes more expensive or even when one lowers the wells' water-table beneath the hazard level and polluted water enters the well's aquifer. These factors would lead to a higher cost because the supplier has to spend more on purifying the water.

It will strengthen this paper to support this solution graphically besides it can be very convenient for many readers to see things graphically. Lastly, but not least, it will expand our understanding of the synthesised solution. The objective of next chapter, chapter 3.2.4, is to review the synthesised solution graphically.

3.2.4 Two-variable-phase-diagram

To confirm and expand our understanding of the synthesised solution it is helpful to find the *two-variable-phase-diagram*[47]. It shall be in the (m, x) space. If equations DC and (E3.15) are total differentiated the motions of variable "x" and "m" is found. In order to ensure the existence of steady state the following condition has to hold,

$$\overset{\circ}{m}_t = 0$$

And

$$\overset{\circ}{x}_t = 0$$

In order to find the *steady state locus*[48] for $\overset{\circ}{m}_t = 0$, equation (E3.6) is substituted for $\overset{\circ}{m}_t$,

[47] Explanation: The *two variable phase diagram* is a tool that is suitable for presentation in dynamic analysis. Further reading about *two variable phase diagram* is given by Chiang (1984. p. 628-638).
[48] Definition: *Steady state locus* is a line or a curve, which is a sample of dots where the variable that it represents does not encourage movement with respect to time. Further reading about the steady state locus or the demarcation curve is given by Chiang (1984. p. 629-631).

$$z_t m_t \left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right) - x_t = 0$$

This locus will be characterised with the sign m. Thus it will from now on be called the m-locus. The solution for the first steady state locus, the m-locus, is written as follow

$$(E3.16) \qquad x_t = z_t m_t \left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right)$$

Equation (E3.16) is the steady state locus in the (m, x) space where $\overset{\circ}{m}_t = 0$, called the m-locus. Figure 3.8, below, is a graphical representation of m-locus. The figure is based on μ = 1.

Figure 3.8

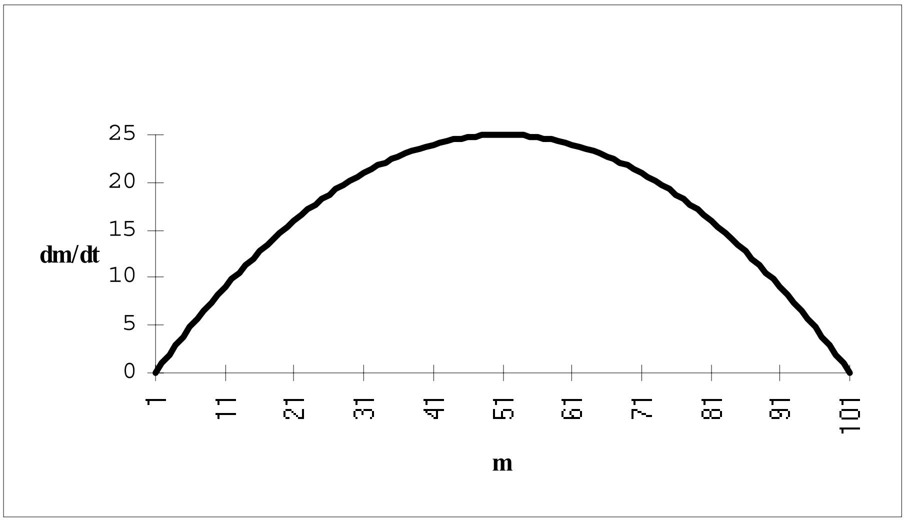

Figure 3.8 shows a two-variable-phase-diagram for *m* and *x*. It shows the m-locus, given the μ = 1.

Two other variations of this locus was drawn in figure 2.8, using the case μ = 0,5 and μ = 5. They are different in shape. As discussed before, only one specific μ is true for one specific aquifer; because each aquifer can be very different from others. Only one shape of this locus is true for one specific aquifer, given that z_t is a known constant. One should be careful with the fact that only the shape of each aquifer is unique and not the size of the locus. From figure 2.9 clearly the rate of precipitation can change the size of the locus.

In order to find the other steady state locus, when $\overset{\circ}{x}_t = 0$, equation (E3.15) needs to be used. Total differentiating (E3.15) with u_{xx} held constant,

$$u_{xx} d\overset{\circ}{x}_t = \left[c_{mm}(F(m_t;\bar{z})) + c_m F_m - u_x F_{mm} - c_m \rho + c_m F_m + c(m_t) F_{mm}\right]dm + \left[u_{xx}(\rho - F_m)\right]dx$$

This locus will be characterised with the sign x. It will from now on be called the x-locus. Now, following assumption is made, $u_x = p$, where p stands for a unit price. Then the second partial derivative of the utility function with respect to supply is zero, $u_{xx} = 0$. The expression above is now changed to

$$0 = \left[c_{mm}(F(m_t;\bar{z})) + c_m F_m - u_x F_{mm} - c_m \rho + c_m F_m + c(m_t) F_{mm}\right]dm$$

From the expression above, it can be concluded that $\overset{\circ}{x}_t$ is not sensitive to "x" extremely close to $\overset{\circ}{x}_t = 0$, it is sensitive to "m". Hence x-locus must be vertical. A specific value of m_t^* is found directly by isolating "m" in equation (E3.15).

Where

m_t^* = The optimal stock of water in the water resource in period t.

$$c_m\left(z_t m_t\left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right)\right) + V_t(\rho - F_m) = 0$$

If equation (E3.7) is substituted for F_m the expression above becomes

$$c_m\left(z_t m_t\left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right)\right) + V_t\left(\rho - z_t\left(1 - (1+\mu)\left(\frac{\psi m_t}{P_t}\right)^{\mu}\right)\right) = 0$$

which can also be written by dropping the two outmost parentheses as,

$$(E3.17) \qquad c_m z_t m_t - c_m z_t\left(\frac{\psi}{P_t}\right)^{\mu} m_t^{1+\mu} + V_t\rho - V_t z_t + V_t z_t(1+\mu)\left(\frac{\psi}{P_t}\right)^{\mu} m_t^{\mu} = 0$$

Equation (E3.17) is the value that determines a vertical position of the steady state locus in the (m, x) space where $\overset{\circ}{x}_t = 0$, called the x-locus. To be able to say something further about m_t^* assumptions for μ have to be made. It is preferable to find one specific value for m_t^*. However this is not so easy for every value of μ. That's why the value $\mu = 1$ has been chosen. So from now on $\mu = 1$. It changes equation (*E*3.17) to

$$(E3.18) \quad V_t(\rho - z_t) + m_t z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) - c_m z_t\left(\frac{\psi}{P_t}\right)m_t^2 = 0$$

(*E*3.18) is a quadratic equation. That's why the solution is found by the quadratic formula (Chiang, 1984, p. 42) which is written in equation (*E*3.19).

$$(E3.19) \quad m_t^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Equation (*E*3.18) substituted for equation (*E*3.19) and then the relationship in square root is simplified,

$$m_t^* = \frac{-z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) \pm \sqrt{\left(z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right)\right)^2 - 4\left(-c_m z_t\left(\frac{\psi}{P_t}\right)\right)V_t(\rho - z_t)}}{2z_t\left(-c_m\left(\frac{\psi}{P_t}\right)\right)}$$

$$m_t^* = \frac{-z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) \pm \sqrt{\left(z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right)\right)^2 + 4\left(c_m z_t\left(\frac{\psi}{P_t}\right)\right)V_t(\rho - z_t)}}{2z_t\left(-c_m\left(\frac{\psi}{P_t}\right)\right)}$$

$$m_t^* = \frac{-z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) \pm \sqrt{(c_m z_t)^2 + 4V_t z_t^2 c_m\left(\frac{\psi}{P_t}\right) + \left(2V_t z_t\left(\frac{\psi}{P_t}\right)\right)^2 + 4V_t z_t c_m\left(\frac{\psi}{P_t}\right)\rho - 4V_t z_t^2 c_m\left(\frac{\psi}{P_t}\right)}}{2z_t\left(-c_m\left(\frac{\psi}{P_t}\right)\right)}$$

$$(E3.20) \quad m_t^* = \frac{-z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) \pm \sqrt{(c_m z_t)^2 + \left(2V_t z_t\left(\frac{\psi}{P_t}\right)\right)^2 + 4V_t z_t c_m\left(\frac{\psi}{P_t}\right)\rho}}{2z_t\left(-c_m\left(\frac{\psi}{P_t}\right)\right)}$$

It can be proved that the last factor of the numerator in equation ($E3.20$) is larger than the other one, meaning that the negative variant of the last factor leads to a negative m_t^* which is not defined and therefore is not an optional solution here. Thus the positive variant of the last factor can only be relevant in this paper because it leads to a positive m_t^*. Furthermore, it proves that there is only one solution for m_t^*, given the assumptions made.

____

Proving that the last factor of the numerator in equation ($E3.20$) is larger than the other part of it,

$$\left| -z_t c_m - z_t 2V_t\left(\frac{\Psi}{P_t}\right) \right| \overset{?}{<} \left| \sqrt{(c_m z_t)^2 + \left(2V_t z_t\left(\frac{\Psi}{P_t}\right)\right)^2 + 4c_m z_t\left(\frac{\Psi}{P_t}\right)V_t\rho} \right|$$

The right-hand side of the expression above can be written as in equation ($E3.21$),

$$(E3.21) \quad \overset{?}{<} \; z_t c_m \sqrt{1 + 4\left(V_t\left(\frac{\Psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\Psi}{c_m z_t P_t}\right)V_t\rho}$$

Or

$$\left| -z_t c_m - z_t 2V_t\left(\frac{\Psi}{P_t}\right) \right| \overset{?}{<} \left| \pm z_t c_m \sqrt{1 + 4\left(V_t\left(\frac{\Psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\Psi}{c_m z_t P_t}\right)V_t\rho} \right|$$

Both sides multiplied with ($1/z_t c_m$),

$$\left| 1 + 2V_t\left(\frac{\Psi}{c_m P_t}\right) \right| \overset{?}{<} \left| \sqrt{1 + 4\left(V_t\left(\frac{\Psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\Psi}{c_m z_t P_t}\right)V_t\rho} \right|$$

Since $c_m < 0$ then following must be true

$$4\left(V_t\left(\frac{\Psi}{c_m P_t}\right)\right)^2 > 0$$

and

$$4\left(\frac{\psi}{c_m z_t P_t}\right)V_t\rho < 0$$

But since

$$\left|4\left(V_t\left(\frac{\psi}{c_m P_t}\right)\right)^2\right| > \left|4\left(\frac{\psi}{c_m z_t P_t}\right)V_t\rho\right|$$

the following must be true

$$(E3.22) \qquad \sqrt{1+4\left(V_t\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)V_t\rho} \quad > \quad 1$$

and the right-hand side must be greater than 1,

$$1 \; < \; \left|\sqrt{1+4\left(V_t\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)V_t\rho}\right|$$

Now the left hand side has to be proven to be less than one and larger than zero, thus:

$$\left|1+2V_t\left(\frac{\psi}{c_m P_t}\right)\right| \; < \; 1$$

Equation ($E3.15$) substituted for V_t (like in chapter 4.2.5),

$$\left|1+2\frac{c_m F(m_t;\bar{z})}{(F_m-\rho)}\left(\frac{\psi}{c_m P_t}\right)\right| \; < \; 1$$

Cancelling the c_m's,

$$\left|1+2\frac{F(m_t;\bar{z})}{(F_m-\rho)}\left(\frac{\psi}{P_t}\right)\right| \; < \; 1$$

By referring to the arguments in chapter 4.2.5 it is quite appropriate to assume the condition above to be true where $F(m_t;\bar{z}) \rightarrow 0$ and (F_m-ρ) ≈ 1, ψ = 1 and P_t is generally a very large positive number. Then following must be true:

$$0 < \quad 1 + 2V_t\left(\frac{\psi}{c_m P_t}\right) < 1$$

which must give

$$\left|1 + 2V_t\left(\frac{\psi}{c_m P_t}\right)\right| \quad < \quad \left|\sqrt{1 + 4\left(V_t\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)V_t\rho}\right|$$

The proof is completed.

Before both steady state locuses are drawn into the same two-variable-phase-diagram, further information is needed. It can be shown that in absense of the stock effect and the discount rate (meaning that the stock effect, c_m, and the discount rate, ρ, are both equal to zero) the optimal solution is m_t^* = m_{MSY}. But when the discount rate, ρ, become positive the optimal solution moves to the left of m_{MSY}, where m_t^* < m_{MSY}. However, in this more specific case, with c_m < 0, it will be proved that the optimal solution moves to the right of the optimal solution m_{MSY}. The comparison between the expressions of m_{MSY} and m_t^* will do that. In order to find m_{MSY} the *stationary point*[49] of the growth model is found because m_{MSY} is the *stationary value*[50], meaning,

$$F_m(m_{MSY}) = 0$$

Equation (E3.7) substituted for the expression above

$$z_t - z_t(1+\mu)\left(\frac{\psi m_{MSY}}{P_t}\right)^{\mu} = 0$$

$-z_t$ added and -1 multiplied to both sided of the expression above,

$$z_t(1+\mu)\left(\frac{\psi m_{MSY}}{P_t}\right)^{\mu} = z_t$$

[49] Definition: A *stationary value* is the value, which the exogenous variable reaches where the slope of the function is zero (Chiang, 1984, p. 236 and p. 317).
[50] Definition: A *stationary point* is the specific value of the exogenous variable, which gives the stationary value (Chiang, 1984, p. 236 and p. 317; Jacques, 1995, p.199 and 277).

If both sides are multiplied by,

$$\frac{1}{z_t(1+\mu)\left(\frac{\psi}{P_t}\right)^{\mu}}$$

the expression becomes

$$m_{MSY}^{\mu} = \frac{1}{(1+\mu)\left(\frac{\psi}{P_t}\right)^{\mu}}$$

Both sides turned into (1/μ)-th power

$$m_{MSY} = \sqrt[\mu]{\frac{1}{(1+\mu)\left(\frac{\psi}{P_t}\right)^{\mu}}}$$

$\left(\frac{P_t}{\psi}\right)^{\mu}$ pulled out from the μ-th root,

$$m_{MSY} = \frac{P_t}{\psi}\sqrt[\mu]{\frac{1}{(1+\mu)}}$$

Substitute $\mu = 1$, which is according to assumption made before and the expression becomes

$$m_{MSY} = \frac{P_t}{2\psi}$$

Now the comparison between m_{MSY} and m_t^* when $\mu = 1$, can take place.

$$\frac{P_t}{2\psi} \overset{?}{<} \frac{-z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) \pm \sqrt{\left(z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right)\right)^2 - 4\left(-c_m z_t\left(\frac{\psi}{P_t}\right)\right)V_t(\rho - z_t)}}{2z_t\left(-c_m\left(\frac{\psi}{P_t}\right)\right)}$$

Multiplying both sides by $(-2z_t c_m \psi/P_t)$

$$-c_m z_t \overset{?}{<} -z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) \pm \sqrt{\left(z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right)\right)^2 - 4\left(-c_m z_t\left(\frac{\psi}{P_t}\right)\right)V_t(\rho - z_t)}$$

$z_t c_m + z_t 2V_t\psi/P_t$ added to both sides

$$-c_m z_t + z_t c_m + z_t 2V_t\left(\frac{\psi}{P_t}\right) \overset{?}{<} \pm\sqrt{\left(z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right)\right)^2 - 4\left(-c_m z_t\left(\frac{\psi}{P_t}\right)\right)V_t(\rho - z_t)}$$

$c_m z_t$ subtracted from $z_t c_m$ on the left-hand side and equation (E3.21) substituted for the right-hand side,

$$z_t 2V_t\left(\frac{\psi}{P_t}\right) \overset{?}{<} \pm z_t c_m\sqrt{1 + 4\left(V_t\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)V_t\rho}$$

Multiplying both sides by $(1/z_t c_m)$

$$2V_t\left(\frac{\psi}{c_m P_t}\right) \overset{?}{<} \sqrt{1 + 4\left(V_t\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)V_t\rho}$$

Since $c_m < 0$, the left-hand side must be negative and equation (E3.22) confirms that the right hand side is greater than 1. Hence following must be true,

$$2V_t\left(\frac{\psi}{c_m P_t}\right) < 0 < 1 < \sqrt{1 + 4\left(V_t\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)V_t\rho}$$

and the right-hand side is larger than the left-hand side, meaning;

$$m_{MSY} < m_t^*$$

Given that every assumption holds, this conclusion means that the $\overset{\circ}{x}_t = 0$ - curve is definitely to the right of the m_{MSY} point in the two-variable-phase-diagrams as shown in figure 3.9. The two-variable-phase-diagram must include the result of this chapter.

Accordingly, there is only one value for m_t^* and it must be in that part of the defined domain of m which is larger than m_{MSY}: $m_t^* \in [m_{MSY}, P_t]$.

Figure 3.9

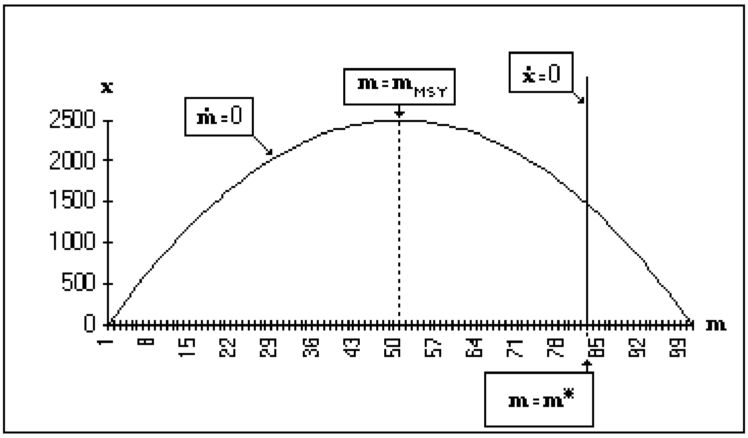

Figure 3.9 shows a two-variable-phase-diagram for variable m and x where m-locus and x-locus are joined together, given the μ =1. The exact location of the m-locus is not known but it must be drawn to right of the m_{MSY}.

Comparing this result with the case where the stock effect and the discount rate is equal to zero, one could say that by taking the pollution vulnerability of the water-resources into account, the optimal stock of water is affected. This vulnerability demands different optimal solution (m_t^*), which is a larger stock of water, "m_t", and a lower supplied quantity of water in each period ("x_t"). Does the partial derivative with respect to stock effect (c_m) confirm this result? It does and the answer is proved and described in chapter 4.1.1.

3.2.5 Describing the dynamic equilibrium

Now lets refresh our memories by recall equation (E3.15), where $1/u_{xx}$ has been multiplied through it,

$$(E3.15) \quad \overset{\circ}{x}_t = \frac{\left[c_m(F(m_t;\bar{z})) + V_t(\rho - F_m)\right]}{u_{xx}}$$

The following analyses is made according to equation (E3.15), which determines the value of the x-locus. If x_t is a positive constant and m_t increases it is clear from equation (E3.15), given the assumption taken in this paper that $\div F_m$ becomes consistently larger, $|c_m|$ consistently smaller and $F(m_t;\bar{z})$ becomes consistently smaller to. Since c_m and F_m is always negative, in $m_t \in [m_{MSY}, P_t]$, and $F(m_t;\bar{z})$ are always positive the negative factor of the expression decreases and the positive one increases. Thus to the right of x-locus, $\overset{\circ}{x}_t$ is positive, $m_t > m_t^* \Rightarrow \overset{\circ}{x}_t > 0$. Accordingly, to the left of the x-locus $\overset{\circ}{x}_t$ is negative, $m_t < m_t^* \Rightarrow \overset{\circ}{x}_t < 0$. Figure 3.10 shows these arguments.

Figure 3.10

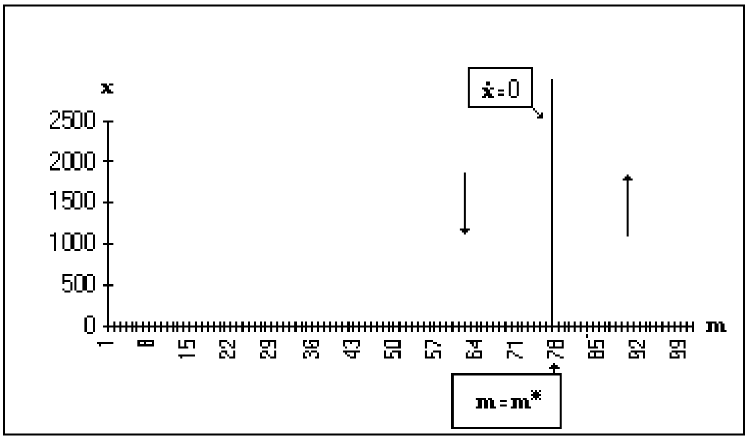

Figure 3.10 shows the motion (dynamic) of x-locus that is positive to the right and negative to the left of it.

Let's refresh our memories again by recall equation (E3.6),

$$(E3.6) \quad (\text{DC}) \quad \overset{\circ}{m}_t = z_t m_t \left(1 - \left(\frac{\psi m_t}{P_t}\right)^{\mu}\right) - x_t$$

Following analysis is made according to equation (E3.6), which determines the value of the m-locus. If m_t is a positive constant and x_t increases it is clear from equation (E3.6), given the assumption taken in this paper that $\overset{\circ}{m}_t < 0$, means that $\overset{\circ}{m}_t$ is negative above the DC curve. If m_t is a positive constant and x_t decreases it is clear from equation (E3.6) that $\overset{\circ}{m}_t > 0$; which means that $\overset{\circ}{m}_t$ is positive under the (E3.6) curve. These motions are drawn in figure 3.11.

Figure 3.11

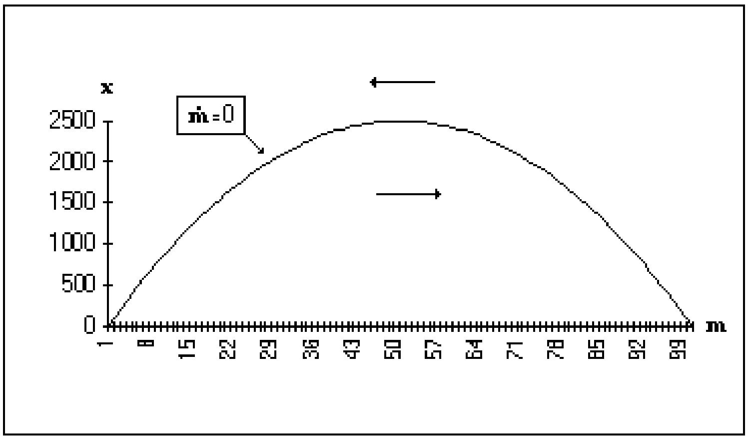

Figure 3.11 shows the motion (dynamic) of m-locus that is positive under and negative above it.

Finally, the motion in the whole system is shown in figure 3.12. It is a combination of figure 3.11 and figure 3.10.

Figure 3.12

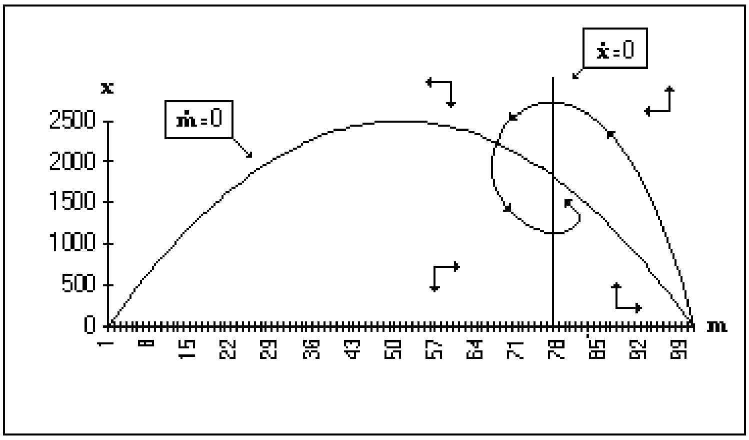

Figure 3.12 is a two-variable-phase-diagram for *m* and *x*. It shows the motion (dynamic) of m-locus and the x-locus in one diagram. From the dynamic described in figure 3.10 and 3.11 the system seems to find its equilibrium where the locuses intersect.

Both m- and x-locuses are plotted in figure 3.12. These combined motions in the system give an opportunity to draw a significant conclusion: Motions of these curves are also indicated. It is clear from the indicated motions that a long run planning will carry the aquifer to (x^*, m^*) which is where these lines' cross. For example if $m_t > m^*$, the strategy is to set the initial x equal to $x_0 > x^*$ and reduce the pumping rate carefully as the water-volume of the aquifer falls to $m_t = m^*$.

Conclusion to problem 3: The following conclusion is drawn from the analyses above: It has been proven that the system has an optimal solution and there exists only one in each period *t*. It is clear from the indicated motions that long run planning will carry the aquifer to equilibrium that is where the lines' cross in figure 3.12. For example if $m_t > m^*$, the strategy is to set the initial x equal to $x_0 > x^*$ and reduce the pumping rate carefully as the water-volume of the aquifer falls to $m_t = m^*$. The factors that influence this equilibrium are the discount rate, ρ, the marginal growth of the aquifer, F_m, and the stock effect, c_m. This corresponds to equation (*E*3.15).

There is a negative relationship between the discount rate, ρ, and the optimal stock of water, $m*$. It describes the community's preference of the water consumption with respect to time. ρ is assumed to be positive. It becomes larger if the community has a larger preference of consumption today than tomorrow. Thus, it will most likely increase in a warm and sunny day when the weather forecast is cloudy for tomorrow. This corresponds to equation (*E*3.15) and supported by a two-variable-phase-diagram in chapter 4.2.2

The marginal growth of the aquifer, F_m, has been discussed in details before. Under natural condition one specific rate of precipitation would create one specific environmental carrying capacity. When the pumping from the aquifer begins, the water volume in the aquifer falls below the environmental carrying capacity ($m_t < P_t$). While the marginal growth is negative (in $m_t > m_{MSY}$) the factor encourages the increase of the pumping rate. While the marginal growth is positive (in $m_t < m_{MSY}$) the factor encourages the reduction of the pumping rate. So, if F_m was the only factor in equation (*E*3.15), it would encourage to keep the aquifer where its vertical and horizontal growth rate is equal. Meaning that the optimal solution would be $m_t^* = m_{MSY}$. It is somewhere between full and empty, which depends on the flow that again, depends on factors like the hydraulic conductivity and the slope of the aquifer. The greater the flow in the aquifer at the P_t *ceteris paribus,* the further m_{MSY} from it. The smaller the flow in the aquifer at the P_t *ceteris paribus,* the nearer m_{MSY} from it. This corresponds to equation (*E*3.15).

The stock effect, c_m, seems to encourage decreased extraction. Recalling that when c_m < 0 and the $F(m_t;\bar{z}) > 0$, the first factor encourages the decrease of pumping from the aquifer. This effect is largest in the first part of the defined domain of m_t ($m_t \in [0, m_{MSY}]$). The stock effect, c_m, requires some amount of water in the aquifer because of the increased costs of the water that was discussed in details before. This corresponds to equation (*E*3.15) and supported by a two-variable-phase-diagram in chapter 4.1.1

3.3. Economic view of the reservoir system

Everything that was discussed and assumed for the social benefit of the groundwater system, chapter 3.2.1, is also assumed to be true for the reservoir system, chapter 3.3. This is proper because it is known that these characters of utility and marginal utility seem to be suitable for a vast variety of goods (Case and Fair, 1989, p. 78; Brown, 1995, pp. 48-49). Also, because in this paper the consumers are assumed to have the same water consumption pattern.

3.3.1. Cost of supplying from the reservoir system

The reservoir system is open because the dam itself is an open lake and some of the entering water is surface water. This means that the reservoir system can be much easier polluted than the ground water system. The wind and the flowing water bring for example sediments, animals' excrement and living creatures bathe in it (Carpenter, 1977, p 431-432; Pleym, 1989, pp. 181-182) which threatens the purity of the water by bacteria (micro-organisms) and thus the community's health (Carpenter, 1977, p. 401; Pleym, 1989, pp. 189-195). Therefore water must be filtered and sterilised before it is supplied to the consumer. It brings costs and thus bacteria inflict cost upon the society.

$$(E3.23) \quad c(b)$$

Where

b = Amount of bacteria

$c'(b) > 0 \quad ; \quad c''(b) > 0$

The water level in the dam influences the water purity. The water has a property of purifying itself (Pleym, 1989, p. 98). The unwanted material that is heavier than the water (Pleym, 1989, pp. 182-183; p. 25) falls to the bottom and keeps most of the flourishing bacteria. Thus the purity of the water gets better as it gets closer to the surface: So, the greater the water volume is, the purer is the water (Pleym, 1989, p. 98).

$(E3.24) \quad b(s_t)$

Where

$b'(s_t) < 0 \quad ; \quad b''(s_t) < 0$

The water temperature also influences the water purity. It is a worse condition for bacteria to flourish in the temperature interval 0 - 10°C than at 10 - 20°C (Carpenter, 1977, pp. 230-231). So, the higher the water's temperature the purer is the water.

$(E3.25) \quad b(v)$

Where

$v =$ Water temperature

$b'(v) > 0 \quad ; \quad b''(v) > 0$

The waters' temperature in Bergen often reaches the higher temperature interval in the summer months (especially the surface water). Since it is cooler in the winter the water's purity is better in the winter than in the summer months.

The water level in the dam influences the water temperature too since it takes more time to warm up a larger volume of water (Otnæs and Ræstad, 1978, p. 145). Therefore the fluctuations of the temperature between days and nights, winters and summers are smaller for a greater volume of water. So, if the water volume in the dam increases, its water gets purer.

$(E3.26) \quad v(s_t)$

Where

$v'(s_t) < 0 \quad ; \quad v''(s_t) < 0$

By imbedding equation (*E*3.26) into equation (*E*3.25) following can be written,

$b(v(s_t))$

Imbedding the expression above into equation (E3.23) it becomes,

$$c = c(b(v(s_t)))$$

Where

$$c'(s_t) < 0 \qquad ; \qquad c''(s_t) > 0$$

The water's taste, smell and appearance could cause increased cost or reduced utility (Pleym, 1989, p. 244). The water level in the dam also influences these factors. Small grains of sediments, rotten organic waste and excrement are good examples of factors that give the water a bad taste. They are heavier than water and they settle into the bottom. The greater the distance between the bottom and the surface of the dam, the better the taste of the water gets as it gets closer to the surface. So, if the dam contains more water, the taste of water is better.

(E3.27) $\quad f(s_t)$

Where

f = Waters "flavour" (taste)

$$f'(s_t) > 0 \qquad ; \qquad f''(s_t) > 0$$

The better the taste of water the higher is the consumer's utility, which can be written as lower social cost of consuming the water as demonstrated in equation (E3.28).

(E3.28) $\quad c = c(f(s_t))$

Where

$$c'(f) < 0 \qquad ; \qquad c''(f) > 0$$

and then

$$c'(s_t) < 0 \qquad ; \qquad c''(s_t) > 0$$

Another environmental threat for many water-resources nowadays is acid rain (Neher, 1990, p. 225). This happens to be a problem in many water-resources in Norway (Pleym, 1989, p. 247) and Pleym (1989, p. 202) claims that Bergen is in that section of Norway that has been most affected by acid rain. Acid rain reduces the water quality and has forced many water suppliers to treat the water, including adding calcium into it.

The conveyance system of the reservoir system is assumed to have the same general character as groundwater systems. Included in the reservoir system is a collection of dams, which have a different altitude, thus different variable cost regarding pumping to the collection tank. Accordingly, the reservoir system has the same direct cost connection between the supplied quantity (x_t) and the stock effect ($m_t \approx s_t$) described in the groundwater system which gives,

$$c = c(x_t, s_t)$$

Where

$$c'(x_t) > 0 \quad ; \quad c''(x_t) = 0$$
$$c'(s_t) < 0 \quad ; \quad c''(s_t) > 0$$

The following equation, (E3.29), contains the characters of the cost function described above. It describes the variable cost of supply from the reservoir system.

$$(E3.29) \qquad c = c(x_t, s_t, b(v(s_t)), f(s_t))$$

The arguments above seem to support the supposition mentioned before about the reservoir system being more vulnerable than the groundwater system because it is an open lake and thus of more highly fluctuating quality. A reduced quality increases the suppliers or the demanders cost. To make the matter simple, the supplier is assumed to keep a constant water quality. Recalling the relation between the water volume and the quality of water; if the water volume in the dam reduces, *ceteris paribus*, then the quality of the water in the dam will reduce. As a result the supplier has to put more chlorine into it to keep it pure enough and clean or replace the filters more often to keep the taste fine.

Taste, temperature and bacteria seem to pull the cost function in the same direction at the same marginal degree as the direct stock influence on social cost, $c(s_t)$. Thus, they can be gathered for simplification. Hence, the cost function above, (E3.29), changes to the following expression

$$(E3.30) \quad c = c(x_t, s_t)$$

Or

$$(C.10) \quad c(x_t, s_t) = c(s_t)x_t$$

Figure 3.13

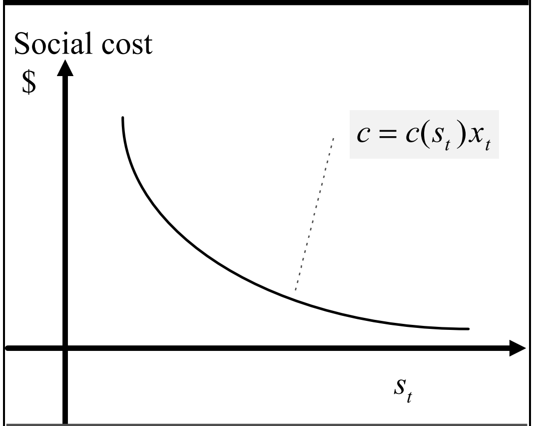

Figure 3.13 shows how the cost function is as a function of the total stock effect. If the volume of saturated zone increases the total stock increases and the total cost falls.

Making a further assumption which are in context with the relevant arguments above,

$$\frac{\partial c(s_t)}{\partial s_t} < 0 \qquad ; \qquad \frac{\partial c^2(s_t)}{\partial s_t^2} \geq 0$$

This means that the cost function $c(s_t)x_t$ is decreasing and convex in s_t. The larger the water stock the lower is the cost (see figure 3.13).

From the discussion above there are many things that support the assumption that the pollution vulnerability is more in the reservoir system than in the groundwater system (Carpenter, 1977, p. 431). If that is true then the following relationship is true,

$$(C.11) \quad \left|\frac{\partial c}{\partial s_t}\right| > \left|\frac{\partial c}{\partial m_t}\right|$$

It could be written as $|c_s| > |c_m|$. This assumption will be used from now on in this paper. It happens to be very useful for comparing some results in later chapters.

As discussed before the following expressions are assumed to be true descriptions of the cost of extracting water (see figure 3.13) which also was assumed by Tsur and Zemel (1995, p. 151) and Neher (1990, p. 261),

$$\frac{\partial c(x_t, s_t)}{\partial x_t} > 0 \qquad ; \qquad \frac{\partial c^2(x_t, s_t)}{\partial x_t^2} = 0$$

The cost function $c(s_t)x_t$ is increasing in x_t.

By collecting central facts and concepts of the reservoir system together and add it into the context of the hydrologic cycle, figure 2.2 changes to figure 3.14.

Figure 3.14

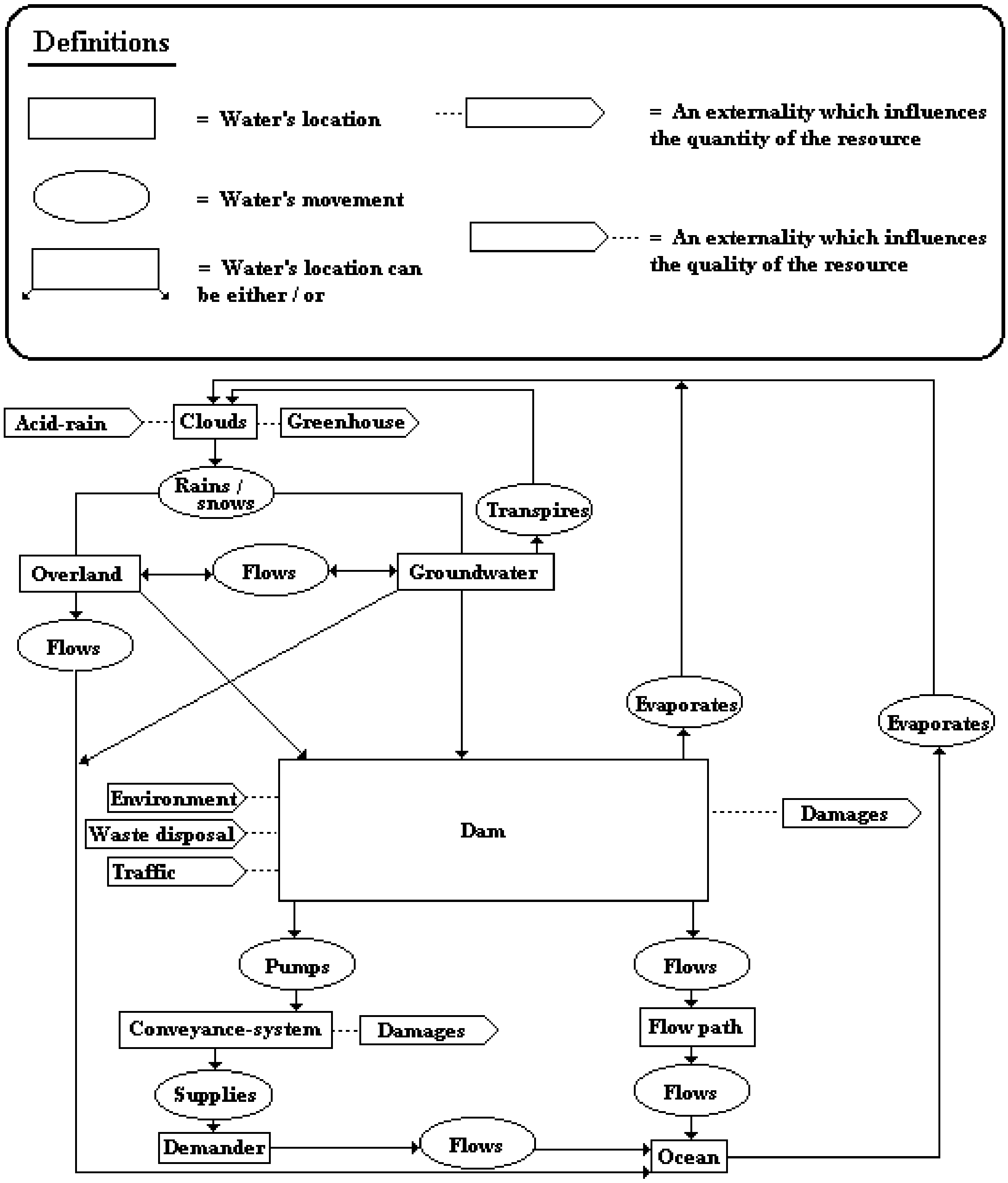

Figure 3.14 shows the context of the hydrologic cycle, given that a dam has been placed into it.

3.3.2. Maximisation of the reservoir system

Figure 3.15 below sums up the characters of the utility function and the cost function into the same diagram.

Figure 3.15

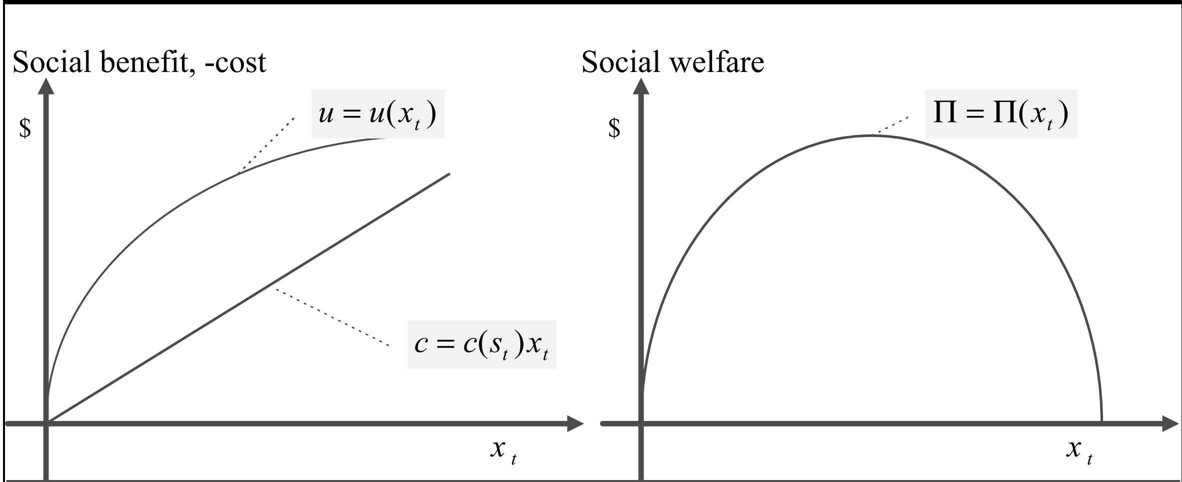

Figure 3.15 shows the social welfare graphically as a function of supplied quantity in the right panel. It shows the social benefit and the social cost together graphically as a function of supplied quantity in the left panel. This is relevant in society which has reservoir water system, which seems to have the same marginal character as in the case of society with a groundwater system.

The ground water- and the reservoir system have been assumed to be equal in all events with two exceptions: $c_s \neq c_m$ and $\mu \neq \sigma$. Thus, the calculation is similar between the cases and the synthesised solution will be equation (*E*3.31).

$$(E3.31) \qquad \overset{\circ}{x}_t u_{xx} = \left[c_s(F(s_t;\bar{z})) + V_t(\rho - F_s)\right]$$

Conclusion to problem 4: Because of this similarity between the groundwater system and the reservoir system the answer to problem 4 is generally the same as to problem 3. The difference between the systems will be clearer in chapter 4.

The following conclusion is drawn from the analyses above: It has been proven that the system has an optimal solution and there exists only one in each period *t*. It is clear from the indicated motions that long run planning will carry the reservoir system to equilibrium that is where these lines' cross in figure 3.9. For example if $s_t > s^*$, the strategy is to set the initial x equal to $x_0 > x^*$ and reduce the pumping rate carefully as the water-volume of the aquifer falls to $m_t = m^*$. The factors that influence this equilibrium are the discount rate, ρ, the marginal growth of the aquifer, F_s, and the stock effect, c_s.

The discount rate, ρ, encourages increase in the extraction from the resource. It describes the community's preference of the water consumption with respect to time. ρ is assumed to be positive. It gets larger if the community has a larger preference of consumption today than tomorrow. Thus, it will most likely increase in a warm and sunny day when the weather forecast for tomorrow is cloudy. This is accordance with equation (*E*3.31) and supported by a two-variable-phase-diagram in chapter 4.2.2

The marginal growth of the aquifer, F_s, has been discussed in details before. Under natural conditions one specific rate of precipitation will create one specific environmental carrying capacity. When the pumping from the dam begins, the water volume falls below the environmental carrying capacity ($s_t < P_t$). While the marginal growth is negative (in $s_t > s_{MSY}$) the factor encourages the increase of the pumping rate. While the marginal growth is positive (in $s_t < s_{MSY}$) the factor encourages the reduction of the pumping rate. So, if F_s was the only factor in equation (*E*3.31), it would encourage to keep the dam where its vertical and horizontal growth rate is equal, meaning that the optimal solution would be $s_t^* = s_{MSY}$. It is somewhere between full and empty, which depends on the outflow, that again depends on e.g. the hydraulic conductivity and the slope of the landscape. The greater the outflow at the P_t *ceteris paribus,* the further s_{MSY} from it. The smaller the outflow in the aquifer at the P_t *ceteris paribus,* the nearer s_{MSY} from it. This corresponds to equation (*E*3.31).

The stock effect, c_s, seems to encourage to decreased extraction. Recalling that $c_s < 0$ and the $F(s_t;\bar{z}) > 0$, the first factor encourages the decrease of pumping from the dam. This effect is largest in the first part of the defined domain of s_t $\left(s_t \in [0, s_{MSY}]\right)$. c_s demands for some amount of water in the aquifer because of the increased costs of the water that was discussed in details before. This corresponds to equation (*E*3.31) and is supported by a two-variable-phase-diagram in chapter 4.1.1. Accordingly, in the case of a reservoir system the size of the stock effect can be seasonal. As said before, the bacteria flourish when the water reaches the summer temperatures.

4 Comparison of models and comparative static results

4.1 Comparing the water systems

There are two main differences that are taken into consideration in comparing the two water systems. They are the ***flow*** and the ***stock effect***. As said before, the ground water- and the reservoir system have been assumed to be equal in all events with two exceptions: $c_s \neq c_m$ and $\mu \neq \sigma$. This comparison gives the solution to *problem 5*.

4.1.1 Influence of the stock effect on the optimal solution.

The stock effect is affected by amongst other things, the system's vulnerability. As was discussed before, it is more likely that a reservoir system is more vulnerable than a groundwater system. Technically this means that $|c_s| > |c_m|$. In order to be able to make a comparison, the following question has to be answered:

The problem of chapter 4.1.1 is: *How does the stock effect, c_m, influence the optimal stock of water, m_t^*, in each period?*

It can be proved that

$$\frac{\partial m_t^*}{\partial c_m} < 0$$

Meaning, that if $|c_m|$ increases, m_t^* increases.

The proof is provided by taking the partial differentiates with respect to the stock effect from equation (*E*3.20) and examining its sign. Since $c_m < 0$ the sign must be negative as noted above. Recall that the every sufficient condition is assumed to hold in this paper.

$$\frac{\partial m_t^*}{\partial c_m} = \frac{\left[-z_t + \frac{1}{2}\left((z_t c_m)^2 + \left(2V_t z_t \frac{\psi}{P_t}\right)^2 + 4c_m z_t \frac{\psi}{P_t} V_t \rho\right)^{-\frac{1}{2}}\left(2z_t^2 c_m + 4z_t \frac{\psi}{P_t} V_t \rho\right)\right]\left[2(-c_m)\frac{\psi}{P_t} z_t\right]}{\left(2z_t\left(-c_m\left(\frac{\psi}{P_t}\right)\right)\right)^2}$$

$$-\frac{-2\frac{\psi}{P_t} z_t\left[-z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) + \sqrt{\left(z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right)\right)^2 - 4\left(-c_m z_t\left(\frac{\psi}{P_t}\right)\right)V_t(\rho - z_t)}\right]}{\left(2z_t\left(-c_m\left(\frac{\psi}{P_t}\right)\right)\right)^2}$$

It is obvious that the denominator is always positive. The second numerator to the expression is also always positive as shown before in the proof of the sign of equation (E3.20).

$$\left[-z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right) - \sqrt{\left(z_t\left(c_m + 2V_t\left(\frac{\psi}{P_t}\right)\right)\right)^2 - 4\left(-c_m z_t\left(\frac{\psi}{P_t}\right)\right)V_t(\rho - z_t)}\right] > 0$$

Now solving the parenthesis in the first numerator

$$2z_t^2 c_m \frac{\psi}{P_t} - \frac{2z_t^3 c_m^2 \frac{\psi}{P_t} + 4V_t \rho c_m z_t^2 \left(\frac{\psi}{P_t}\right)^2}{c_m z_t \sqrt{1 + 4\left(\frac{V_t \psi}{c_m P_t}\right)^2 + \frac{4V_t \psi \rho}{c_m P_t}}}$$

Let us denote

$$\gamma = \sqrt{1 + 4\left(\frac{V_t \psi}{c_m P_t}\right)^2 + \frac{4V_t \psi \rho}{c_m P_t}}$$

and the expression becomes

$$2z_t^2 c_m \frac{\psi}{P_t} - \frac{2z_t^3 c_m^2 \frac{\psi}{P_t} + 4V_t \rho c_m z_t^2 \left(\frac{\psi}{P_t}\right)^2}{c_m z_t \gamma}$$

Cancelling $c_m z_t$ from the two last factors of the first numerator

$$2z_t^2 c_m \frac{\Psi}{P_t} - \frac{2z_t^2 c_m \frac{\Psi}{P_t} + 4V_t \rho z_t \left(\frac{\Psi}{P_t}\right)^2}{\gamma}$$

then split up the last factor

$$2z_t^2 c_m \frac{\Psi}{P_t} - \frac{2z_t^2 c_m \frac{\Psi}{P_t}}{\gamma} - \frac{4V_t \rho z_t \left(\frac{\Psi}{P_t}\right)^2}{\gamma}$$

The first and the third factors of the expression are negative, recalling from (E3.22) that $\gamma > 1$ and $c_m < 0$. Then the absolute value of the first factor must be larger than the absolute value of the second factor and the whole expression is proven to be negative.

Since the first numerator of the partial differentiates with respect to the stock is negative and the second numerator is positive we still have to prove that their sum is negative. Solving the parentheses to the second numerator and starting the comparison between the numerators

$$-2z_t^2 c_m \frac{\Psi}{P_t} - 4V_t \left(\frac{z_t \Psi}{P_t}\right)^2 + 2z_t^2 c_m \frac{\Psi}{P_t}\gamma$$

Adding it to the first numerator, where it must be less than zero

$$2z_t^2 c_m \frac{\Psi}{P_t} - \frac{2z_t^2 c_m \frac{\Psi}{P_t}}{\gamma} - \frac{4V_t \rho z_t \left(\frac{\Psi}{P_t}\right)^2}{\gamma} - 2z_t^2 c_m \frac{\Psi}{P_t} - 4V_t \left(\frac{z_t \Psi}{P_t}\right)^2 + 2z_t^2 c_m \frac{\Psi}{P_t}\gamma \overset{?}{<} 0$$

Cancelling the first and the fourth factor

$$-\frac{2z_t^2 c_m \frac{\Psi}{P_t}}{\gamma} - \frac{4V_t \rho z_t \left(\frac{\Psi}{P_t}\right)^2}{\gamma} - 4V_t \left(\frac{z_t \Psi}{P_t}\right)^2 + 2z_t^2 c_m \frac{\Psi}{P_t}\gamma \overset{?}{<} 0$$

γ multiplied through the expression

$$-2z_t^2 c_m \frac{\psi}{P_t} - 4V_t\rho z_t\left(\frac{\psi}{P_t}\right)^2 - 4V_t\left(\frac{z_t\psi}{P_t}\right)^2\gamma + 2z_t^2 c_m\frac{\psi}{P_t}\gamma^2 \overset{?}{<} 0$$

P_t/ψ multiplied through the expression

$$-2z_t^2 c_m - 4V_t\rho z_t\left(\frac{\psi}{P_t}\right) - 4V_t z_t^2\left(\frac{\psi}{P_t}\right)\gamma + 2z_t^2 c_m\gamma^2 \overset{?}{<} 0$$

Collecting common factors together and the expression becomes

$$2z_t^2 c_m\left[\gamma^2 - 1\right] - 4V_t z_t\left(\frac{\psi}{P_t}\right)\left[\rho + z_t\gamma\right] < 0$$

Since c_m is negative and γ larger than one it is clear from the expression above that the sum of the numerators to the partial differentiates with respect to the stock is negative. Where the denominator is positive, the whole partial differentiates with respect to the stock must be negative, meaning

$$\frac{\partial m_t^*}{\partial c_m} < 0$$

The proof is completed.

Figure 4.1

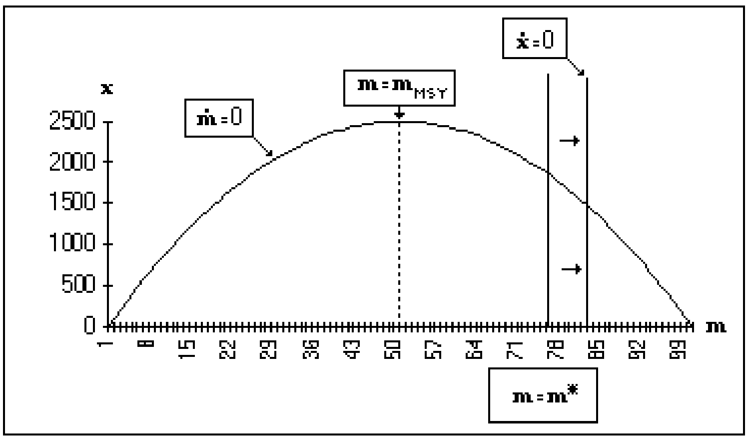

Figure 4.1 is a two-variable-phase-diagram for *m* and *x*. It shows in which direction the x-locus moves in a two-variable-phase-diagram if the stock effect increases.

The conclusion to the chapter 4.1.1: If every assumption holds then it has been proven that if the stock effect (∂c/∂m) increases then m_t^* must increase also, *ceteris paribus* (see figure 4.1). It means that if each litre of water taken from the water-resource gives greater marginal effect to the social cost then it will be feasible to keep higher water level in the water-resource. This is in accordance with $(E3.15)$.

As was discussed before, it is most likely that a reservoir system is more vulnerable than a groundwater system. Technically this means that $|c_s| > |c_m|$. If this is true then it is possible to make a comparison between the societies. The stock effect will demand greater stock of water in the reservoir system than in the groundwater system, relatively (or absolutely if the systems size is approximately equal), *ceteris paribus*. This will be used to solve problem 5.

4.1.2 Influence of flow on the optimal solution.

The flow represents only the outflow of the water-resource as stressed beneath figure 2.5 in chapter 2.4.3. Since location of the dam is especially chosen for its purpose the soil is probably tested for, e.g. leakage. The wall of the dam is specially constructed including the consistence. Thus a reservoir system is assumed to have less proportional seepage than a groundwater system, *ceteris paribus*. However this does not always have to be true even though it is most appropriate to assume in the context of this paper, because of the geological circumstances in Reykjavik that makes the aquifer especially porous and Bergen, that is surrounded by especially tight rock. This relation would not be true e.g. when the soil is so tight that the hydraulic conductivity gets very low or the slope of the aquifer is not so steep. Hence, it is most appropriate to assume greater flow in the groundwater system than in the reservoir system, meaning σ > μ. In order to be able to make a comparison, the following question has to be answered:

This is the problem of chapter 4.1.2: *How does aquifer's different ability to flow influences the optimal stock of water (m_t^*)?*

As discussed before, μ = 2, means another aquifer that has a slower flow than an aquifer with μ = 1. This assumption, μ = 2, equation (*E*3.17) changes to

$$V_t(\rho - z_t) + c_m z_t m_t + 3V_t z_t\left(\frac{\Psi}{P_t}\right)^2 m_t^2 - c_m z_t\left(\frac{\Psi}{P_t}\right)^2 m_t^3 = 0$$

It is shown in appendix 6.1, by a numerical solution that an increased μ leads to a larger value of m_t^*. It means that it will not be feasible to lower one aquifer's water stock, m_t^*, as much as one's with a faster groundwater flow.

Conclusion to problem of chapter 4.1.2: If every assumption holds then it is shown by a numerical simulation in appendix 6.1 that the smaller flow in the aquifer will lead to a larger m_t^*, *ceteris paribus*. The flow is dependent on the hydraulic conductivity and the slope of the aquifer. In other words, it is optimal to keep small stock of water in the defined water-resource if the ability of flow in one specific water-resource is high. This corresponds to equation (*E*3.15).

As was discussed before, it is most likely that there is greater flow in the groundwater system than in the reservoir system. Technically this means that $\sigma > \mu$. If this is true then it is possible to make a comparison between the systems. Slower flow will demand greater stock of water in the reservoir system than in the groundwater system, relatively (or absolutely if the systems size is approximately equal), *ceteris paribus*. In other words, it will not be feasible to lower the water stock in a reservoir system, s_t^*, as much as in a groundwater system m_t^*. This will be used to solve problem 5.

Conclusion to problem 5: From the analyses of chapter 4.1.1 and 4.1.2, the following conclusion is drawn. By isolating these two main differences between the two systems, both seem to demand relatively greater stock of water in the reservoir system than in the groundwater system, *ceteris paribus*. The reasons are; it has been argued in this paper that the groundwater system has greater outflow of water and is not as vulnerable to pollution as the reservoir system.

This comparison has two other interesting aspects. These are **seasonal** and **geographical**. It has been claimed in this paper that waters temperature influences both the stock effect and the flow. The stock effect will be effected via the population of bacteria. If water temperature declines the water becomes purer. This will be especially favourable for the reservoir system where the water in the ground water system is assumed to be able pure itself anyhow. So here the c_s will be lowered and the c_m remains the same. The flow will be effected via the hydraulic conductivity. Given that water temperature never reaches beneath 4°C, a reduction of it will reduce the outflow. Since the outflow of the groundwater is totally dependent to the flow underground but the reservoir system only partially this will have bigger effect on μ than σ.

The **seasonal** aspect is that the difference of the water systems in Bergen and Reykjavik with respect to relative optimal stock of water seems to decrease in the winter and increase in the summer, *ceteris paribus*.

Accordingly, the **geographical** aspect is that the nearer the society is to the Equator the greater the difference of optimal stock of water between a ground water – and reservoir system, *ceteris paribus*, meaning that the nearer the society is to the Equator the greater is the stimulus to run a ground water system than reservoir system.

4.2 Other comparative static results

4.2.1 What influence does price have on the optimal solution?

Following is the problem of chapter 4.2.1,

How will price influence m_t^* *?*

Furthermore it is very interesting to discuss the effects that the price will have on m_t^* given the defined circumstances. It can be proved that the relationship is negative between the price and the optimal stock of water, m_t^*, in a given water resource.

The proof is provided by taking the partial differentiates with respect to the stock effect from equation (*E*3.20) and examining its sign. Recalling that the every sufficient condition is assumed to hold in this paper.

As shown before in equation (*E*3.20), m_t^* can be written as

$$m_t^* = \frac{-z_t c_m - z_t 2V_t\left(\frac{\Psi}{P_t}\right) + \sqrt{(z_t c_m)^2 + 4\left(V_t z_t\left(\frac{\Psi}{P_t}\right)\right)^2 + 4\left(c_m z_t\left(\frac{\Psi}{P_t}\right)V_t \rho\right)}}{-2z_t c_m\left(\frac{\Psi}{P_t}\right)}$$

$$m_t^* = \frac{-z_t c_m - z_t 2(p - c(m_t))\left(\frac{\Psi}{P_t}\right) + \sqrt{(z_t c_m)^2 + 4\left((p - c(m_t))z_t\left(\frac{\Psi}{P_t}\right)\right)^2 + 4\left(c_m z_t\left(\frac{\Psi}{P_t}\right)(p - c(m_t))\rho\right)}}{-2z_t c_m\left(\frac{\Psi}{P_t}\right)}$$

Then the partial differentiate with respect to price is taken, and its sign examined.

$$\frac{\partial m_t^*}{\partial p} = \frac{-z_t 2\left(\frac{\Psi}{P_t}\right) + 1/2(Þ)^{-1/2}\left(8\left((p - c(m_t))z_t\left(\frac{\Psi}{P_t}\right)\right)z_t\left(\frac{\Psi}{P_t}\right) + 4c_m z_t\left(\frac{\Psi}{P_t}\right)\rho\right)}{\left(-2z_t c_m\left(\frac{\Psi}{P_t}\right)\right)}$$

Where

$$Þ = (z_t c_m)^2 + 4\left((p - c(m_t))z_t\left(\frac{\Psi}{P_t}\right)\right)^2 + 4\left(c_m z_t\left(\frac{\Psi}{P_t}\right)(p - c(m_t))\rho\right)$$

$$\frac{\partial m_t^*}{\partial p} = \frac{-z_t 2\left(\frac{\Psi}{P_t}\right) + (Þ)^{-1/2}\left(4(p - c(m_t))\left(z_t\left(\frac{\Psi}{P_t}\right)\right)^2 + 2c_m z_t\left(\frac{\Psi}{P_t}\right)\rho\right)}{\left(-2z_t c_m\left(\frac{\Psi}{P_t}\right)\right)}$$

The common factor, $z_t 2\left(\frac{\Psi}{P_t}\right)$, cancelled

$$\frac{\partial m_t^*}{\partial p} = \frac{-1 + (Þ)^{-1/2}\left(2(p - c(m_t))\left(z_t\left(\frac{\Psi}{P_t}\right)\right) + c_m\rho\right)}{-c_m}$$

or

$$\frac{\partial m_t^*}{\partial p} = \frac{-1 + \dfrac{2(p - c(m_t))\left(z_t\left(\frac{\Psi}{P_t}\right)\right) + c_m\rho}{\sqrt{(z_t c_m)^2 + 4\left((p - c(m_t))z_t\left(\frac{\Psi}{P_t}\right)\right)^2 + 4\left(c_m z_t\left(\frac{\Psi}{P_t}\right)(p - c(m_t))\rho\right)}}}{-c_m}$$

If the context $(C.9)$ holds

$$(C.9)\quad \sqrt{(z_t c_m)^2 + 4\left((p - c(m_t))z_t\left(\frac{\Psi}{P_t}\right)\right)^2 + 4\left(c_m z_t\left(\frac{\Psi}{P_t}\right)(p - c(m_t))\rho\right)} > 2(p - c(m))\left(z_t\left(\frac{\Psi}{P_t}\right)\right) + c_m\rho$$

then the relationship between optimal stock of water, m_t^*, and price, p, is negative, or

$$\frac{\partial m_t^*}{\partial p} < 0 .$$

Equation ($E3.21$) substituted for the left side of ($C.9$),

$$z_t c_m \sqrt{1+4\left((p-c(m_t))\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)(p-c(m_t))\rho} \overset{?}{>} 2(p-c(m_t))\left(z_t\left(\frac{\psi}{P_t}\right)\right) + c_m \rho$$

Both sides multiplied with $1/z_t c_m$,

$$\sqrt{1+4\left((p-c(m_t))\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)(p-c(m_t))\rho} \overset{?}{>} \frac{2(p-c(m_t))\left(\frac{\psi}{P_t}\right)}{c_m} + \frac{\rho}{z_t}$$

The left-hand side is greater than 1 according to equation ($E3.22$). Since $c_m < 0$, following statement about the first factor of the right-hand side must be true:

$$-\infty < \frac{2(p-c(m_t))\left(\frac{\psi}{P_t}\right)}{c_m} < 0$$

Since $z_t = 1$ and $\rho < 1$ but closer to zero, following statement about the second factor of the right-hand side must be true:

$$0 < \frac{\rho}{z_t} < 1 .$$

Thus the right-hand side is smaller than 1 and following is true.

$$\sqrt{1+4\left((p-c(m_t))\left(\frac{\psi}{c_m P_t}\right)\right)^2 + 4\left(\frac{\psi}{c_m z_t P_t}\right)(p-c(m_t))\rho} > \frac{2(p-c(m_t))\left(\frac{\psi}{P_t}\right)}{c_m} + \frac{\rho}{z_t}$$

The conclusion is:

$$\frac{\partial m_t^*}{\partial p} < 0$$

The expression, $\frac{\partial m_t^*}{\partial p}$, has now been proved to be always negative.

The proof is completed.

Figure 4.2

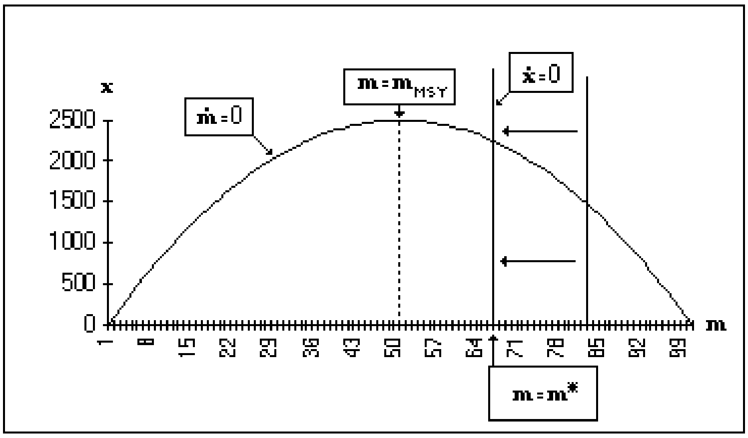

Figure 4.2 is a two-variable-phase-diagram for *m* and *x*. It shows in which direction the x-locus moves in two variable phase-diagrams if the price increases.

Conclusion to the problem of chapter 4.2.1: If all assumptions are true then it is proven above that if the price rises the optimal water stock of the defined water-resource will fall, *ceteris paribus* (see figure 4.2).

There can be many reasons for the growing price of water in those "lucky" regions. Increased demand is one of the most obvious reasons as a result of for example a growing water-intensive industry. This growth can come from e.g. easier access to the global water-market; which provides more customers thus greater demand of water. This happens to be the case for rather new developed industries in Iceland and Norway that aims especially at water-market overseas. By other means, it has been pointed out that the demand for fresh water will increase rapidly through the end of the century (Hamrin, 1983, p. 39). Furthermore, there are many signs for increasing scarcity of water in the world (Bhaskar and Glyn, 1995, p. 19).

From this conclusion it is obvious that if the price falls then the m_t^* moves toward m_t

”full” or P_t. Hence, if the price is low enough, approximately zero, m_t^* will be equal to m_t “full”.

4.2.2 The effect of a change of the social discount rate.

Following is the problem of chapter 4.2.2,

How would the optimal solution, m_t^*, *change if the demanders preferred the water today to the water tomorrow?*

How will m_t^* change if the consumers prefer water today to water tomorrow? This could happen in a warm and sunny season. By taking the partial derivative of the optimal volume of water in the aquifer, m_t^*, with respect to the discount rate, ρ, and examine its sign will answer the question. How does $\partial m_t^* / \partial \rho$ look like? Would it be positive, negative or zero? It is possible to prove that $\partial m_t^* / \partial \rho$ is always negative.

The proof is provided by taking the partial differentiates with respect to discount rate from equation (*E*3.20) and examining its sign. Recall that the every sufficient condition is assumed to hold in this paper.

$$\frac{\partial m_t^*}{\partial \rho} = \frac{\frac{1}{2}\left((z_t c_m)^2 + \left(2V_t z_t \frac{\psi}{P_t}\right)^2 + 4c_m V_t z_t \frac{\psi}{P_t}\rho\right)^{-\frac{1}{2}}\left(4c_m V_t z_t \frac{\psi}{P_t}\right)}{2(-c_m)\frac{\psi}{P_t} z_t}$$

or

$$\frac{\partial m_t^*}{\partial \rho} = \frac{\left(4c_m V_t z_t \frac{\psi}{P_t}\right)}{4(-c_m)\frac{\psi}{P_t} z_t \sqrt{(z_t c_m)^2 + \left(2V_t z_t \frac{\psi}{P_t}\right)^2 + 4c_m V_t z_t \frac{\psi}{P_t}\rho}}$$

or even

$$\frac{\partial m_t^*}{\partial \rho} = \frac{V_t}{-\sqrt{(z_t c_m)^2 + \left(2V_t z_t \frac{\psi}{P_t}\right)^2 + 4c_m V_t z_t \frac{\psi}{P_t}\rho}} < 0$$

The expression ($\partial m_t^* / \partial \rho$) has now been proved to be always negative.

The proof is completed.

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Figure 4.3

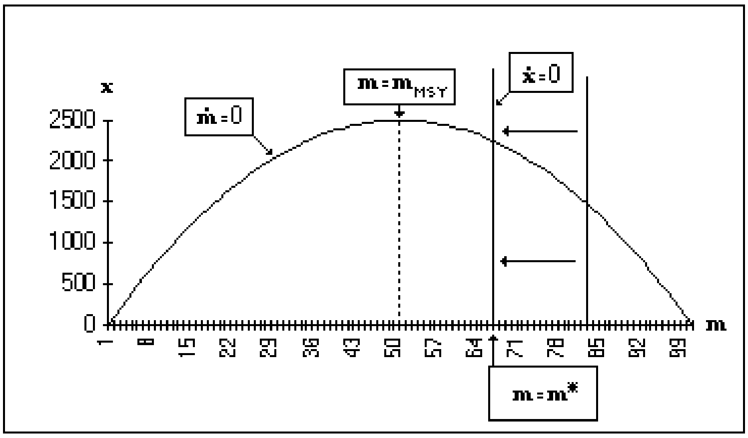

Figure 4.3 is a two-variable-phase-diagram for *m* and *x*. It shows in which direction the x-locus moves in two variable phase-diagram if the ρ increases.

If ρ is initially equal to zero but suddenly becomes positive ($\rho > 0$) then the x-locus moves to the left in the phase-diagram (see figure 4.3). Thus, it is possible to have the steady state solution $m_t^* < m_{MSY}$. This means that when the consumers prefer the water today over the water tomorrow ρ becomes positive and m_t^* moves to the left ("m" decreases) as shown in figure 4.3. When these preferences increase *ceteris paribus*, m_t^* moves further to the left.

Conclusion to problem of chapter 4.2.2: If every assumption is true then it is proven above that if the discount rate increases then m_t^* must decrease, *ceteris paribus*. In a dry season the discount rate increases which reduces the optimal stock of water in the aquifer.

4.2.3 Influence of precipitation on the optimal solution.

How will precipitation influence the optimal solution of the state variable, m_t^*?

$$\partial m_t^* / \partial z_t$$

The relationship between a dry season (i.e. month) and its influence on the water level in a saturated zone is fairly well known. It is especially dependent on the water system's ability to store and transmit water. This ability is very different between water systems. The size of a water system (in m^3) and a relevant rainfall zone (in m^2) are very significant factors. The size of the water system could be changed, but it is not so easy to change the size of the rainfall zone. The formation of the landscape plays a role both to the size of the water system and the rainfall zone. The higher temperature and the number of sunny days would increase the evaporation from the water system, especially in the reservoir system. Dudley (1992, p. 767) also stresses that we could easily have a very fluctuating and uncertain supply of water over time.

4.2.4 Quality of the pumping cone.

What influences would changes in storativity, S, have on changes in the drawdown (η_0-η)? The first partial derivative gives us:

$$\frac{\partial(\eta_0 - \eta)}{\partial S} = -\frac{1}{S} \bullet \frac{0{,}99889Q}{4\pi T}$$

$$\frac{\partial(\eta_0 - \eta)}{\partial S} = -\frac{0{,}99889Q}{4\pi TS} < 0$$

This means that higher storativity will reduce the influence from the pumping rate on the well's drawdown. As mentioned before, a high head in a saturated aquifer will give a higher storativity. High head in a saturated aquifer means that the well's water level is high. Therefore, if the well's water level is high, each litre taken from the well has less influence on the drawdown than if the well's water level was lower.

What influences would changes in transmissivity, T, have on changes in the drawdown (η_0-η)? The first partial derivative gives us:

$$\frac{\partial(\eta_0 - \eta)}{\partial T} = -\frac{1}{T^2}\frac{0{,}99889Q}{4\pi}\ln\frac{2{,}25Tt}{Sr^2} + \frac{0{,}99889Q}{4\pi T}\frac{1}{T}$$

This is a more complicated relationship. It is rearranged to:

$$\frac{\partial(\eta_0 - \eta)}{\partial T} = \frac{0{,}99889Q}{4\pi T^2}\left(1 - \frac{\ln 2{,}25Tt}{Sr^2}\right)$$

From the expression it is obvious that if T is very small, there is a positive influence on the drawdown. By each bigger T the influence is declining. After a while the influence gets negative (increasingly). Then the T is recharging the well that reduces the affects from the pumping on the drawdown. Recalling that there are small values of S and r and large values of T and t it is quite appropriate to assume $\partial(\eta_0-\eta)/\partial T < 0$. Recalling that higher head increases the S and the T, we can say higher head reduces the pumping influence on the drawdown.

4.2.5 Shadow price.

According to Neher (1990, p.168), shadow price is equal to V_t when $\rho \neq 0$ if the optimal solution is reached. According to equation (E3.15) the value of V_t in steady state is found by following.

$$V_t = \frac{c_m(F(m_t;\bar{z}))}{(F_m - \rho)}$$

It is obvious from the results of chapter 3.2.4 that F_m is always a negative number, a very small one when its close to m_{MSY} and bigger when close to P_t. Then clearly V_t is a positive number since c_m is always negative and $F(m_t;\bar{z})$ always positive. When m_t is moving from m_{MSY} to P_t the $c_m \rightarrow 0, F(m_t;\bar{z}) \rightarrow 0$ and $F_m \rightarrow \div 1$. This means that when m_t is moving from m_{MSY} to P_t that the numerator tends to zero while the denominator tends to $1 + \rho$. While the $0 \leq \rho \leq 1$. It has been argued that the discount rate shall reflect social rate of time preference. Meaning that $\rho \approx 0$ if there were full equality between generations, assuming that there will be an apocalypse and $\rho \approx 1$ if there won't be any future for the future generations. So here we will assume ρ close to zero. Accordingly, when m_t is moving from m_{MSY} to P_t the $V_t \rightarrow 0$.

What can we say about the value of V_t in Reykjavik and Bergen? Generally the domestic demand of water is easily met. When it is so, V_t is a very small number because the steady state solution is very near P_t. Besides it has been argued that ρ reflects e.g. peoples belief. If people generally believe that water is inexhaustible even though it is not so ρ tends to zero. Since this paper handles the domestic water-market without any possibilities for access to the global water-market it is easy to claim V_t as a small number. However, this conclusion becomes questionable in chapter 4.2.1.

5 Summary and concluding remarks.

Two water systems are described in chapter 2. These are the ground water system and the reservoir system. Those systems are water resources in two separate communities. There are many factors significant for their growth e.g. their own size. This is because of the different ability of the systems to collect water and make sufficient impedance to hold it. These two abilities are found to be necessary and sufficient condition for to be able to define water a renewable resource. The collection ability in the ground water system is the soils' different ability to absorb water. Its impedance ability is the different ability to transmit water, i.e. porosity of the soil. The collection ability of the reservoir system is rather constant, dependent of the form of the landscape around the dam. Its impedance ability is dependent of the different porosity of the soil and the height of the dam wall. The systems abilities are used to find relevant growth function for the water resources that become significant dynamic constraints in chapter 3.

In chapter 3 both constraints are used to describe and educe two separate optimal solutions, one for each water system, and suggest the appropriate management. Several factors that influence the optimal solution are also discussed. The systems have many things in common. These are e.g. the eduction and the description of the optimal solution. Both systems have one unique optimal solution respectively, m^* and s^*. The suggestion for suitable management is similar. Let us assume that the water stock is larger than the optimal water stock, $m_t > m^*$ and/or $s_t > s^*$. The suggested management is then to set the initial pumping rate, x, above the optimal pumping rate, $x_0 > x^*$ and reduce it carefully to the optimal rate, x^*, as the water-volume of the aquifer falls to $m_t = m^*$ and/or $s_t = s^*$.

The factors that influence this optimal solution are the discount rate, ρ, the marginal growth of the resource, F_m, and the stock effect, c_m. The discount rate, ρ, reflects the community's preference of the water consumption with respect to time. It becomes larger if the community has higher preference of consumption today than tomorrow. The relationship between the discount rate and the optimal water stock is negative in both systems, $\partial m^*/\partial\rho < 0$ and $\partial s^*/\partial\rho < 0$. This means that if the preferences increase it is optimal for the community to reduce the water stock.

The marginal growth of the aquifer, F_m, is a combination of the systems marginal ability to collect and transmit water. It is positive when the marginal ability to collect is larger than marginal ability to transmit, zero when they are even and otherwise negative. The marginal growth of the resource tends to be positive when the stock is small and negative when it is large. The relationship between the marginal growth and the optimal water stock is positive in both systems, $\partial m^*/\partial F_m > 0$ and $\partial s^*/\partial F_s > 0$. This means that if marginal growth is negative then it is optimal for the community to reduce the water stock.

The stock effect, c_m, is the marginal cost with respect to stock of water. This effect is always negative, because when the stock of water falls, its quality does too. If the quality falls then the water needs more purification, involving higher cost. Thus the stock effect enlarges when stock of water falls. Thus, one could say that the stock effect represents one aspect of water resources vulnerability. The relationship between the stock effect and the optimal water stock is positive in both systems, $\partial m^*/\partial c_m < 0$ and $\partial s^*/\partial c_s < 0$. This means that if stock effect increases, it is optimal for the community to increase the water stock.

The systems comparison reveals a strong implication towards higher stock effect and stronger impedance, or less ability to create flow, in the reservoir system than in the ground water system. It was argued that the reservoir system has generally stronger impedance than the ground water system because of the tight dam wall and the location of the dam is specially chosen regarding e.g. soil porosity. Stronger impedance keeps the marginal growth of the resource longer positive with respect to stock of water. It surely motivates to higher optimal stock of water. It was also argued that the reservoir system has larger stock effect because it is more open and thus more vulnerable for external pollution than the ground water system. It also motivates to higher optimal stock of water. Because of the difference in ability to create flow (impedance) and stock effect it is a reason to claim that a reservoir system is proportionally less flexible than a groundwater system. A higher water price is one of the challenges to the flexibility of the water system.

6 Appendix.

6.1 Simulated influence of different μ on the optimal solution.

The following table contains values from a simulation of equation (*E*3.17) done in Excel 4.0. There is a comparison among three values of μ. These are the values 1,2 and 3. The endogenous values (m_t) are integers among 0 to 100 with these included.

Exogenous Value (*m*)	Endogenous Value when μ =1	Endogenous Value when μ =2	Endogenous Value when μ =3
0	-1982	-1982	-1982
1	-2043,05	-2081,45	-2081,99
2	-2102,2	-2179,76	-2181,96
3	-2159,45	-2276,88	-2281,85
4	-2214,8	-2372,76	-2381,65
5	-2268,25	-2467,33	-2481,3
6	-2319,8	-2560,56	-2580,78
7	-2369,45	-2652,37	-2680,03
8	-2417,2	-2742,72	-2779,01
9	-2463,05	-2831,56	-2877,69
10	-2507	-2918,83	-2976
11	-2549,05	-3004,47	-3073,9
12	-2589,2	-3088,43	-3171,33
13	-2627,45	-3170,66	-3268,25
14	-2663,8	-3251,1	-3364,59
15	-2698,25	-3329,7	-3460,3
16	-2730,8	-3406,41	-3555,31
17	-2761,45	-3481,17	-3649,57
18	-2790,2	-3553,92	-3743
19	-2817,05	-3624,62	-3835,54
20	-2842	-3693,2	-3927,13
21	-2865,05	-3759,62	-4017,68
22	-2886,2	-3823,82	-4107,14
23	-2905,45	-3885,74	-4195,42
24	-2922,8	-3945,33	-4282,44
25	-2938,25	-4002,55	-4368,13
26	-2951,8	-4057,32	-4452,4
27	-2963,45	-4109,61	-4535,18
28	-2973,2	-4159,35	-4616,37
29	-2981,05	-4206,49	-4695,9
30	-2987	-4250,98	-4773,66
31	-2991,05	-4292,75	-4849,57
32	-2993,2	-4331,77	-4923,53

Exogenous Value (m)	Endogenous Value when μ =1	Endogenous Value when μ =2	Endogenous Value when μ =3
33	-2993,45	-4367,98	-4995,45
34	-2991,8	-4401,31	-5065,24
35	-2988,25	-4431,72	-5132,78
36	-2982,8	-4459,15	-5197,98
37	-2975,45	-4483,54	-5260,74
38	-2966,2	-4504,85	-5320,95
39	-2955,05	-4523,02	-5378,5
40	-2942	-4538	-5433,28
41	-2927,05	-4549,73	-5485,18
42	-2910,2	-4558,15	-5534,08
43	-2891,45	-4563,22	-5579,88
44	-2870,8	-4564,87	-5622,44
45	-2848,25	-4563,06	-5661,65
46	-2823,8	-4557,73	-5697,39
47	-2797,45	-4548,82	-5729,54
48	-2769,2	-4536,29	-5757,96
49	-2739,05	-4520,08	-5782,52
50	-2707	-4500,13	-5803,11
51	-2673,05	-4476,38	-5819,58
52	-2637,2	-4448,8	-5831,8
53	-2599,45	-4417,31	-5839,63
54	-2559,8	-4381,87	-5842,94
55	-2518,25	-4342,43	-5841,57
56	-2474,8	-4298,92	-5835,4
57	-2429,45	-4251,3	-5824,28
58	-2382,2	-4199,51	-5808,05
59	-2333,05	-4143,49	-5786,57
60	-2282	-4083,2	-5759,68
61	-2229,05	-4018,57	-5727,25
62	-2174,2	-3949,56	-5689,1
63	-2117,45	-3876,11	-5645,08
64	-2058,8	-3798,17	-5595,03
65	-1998,25	-3715,67	-5538,79
66	-1935,8	-3628,57	-5476,2
67	-1871,45	-3536,82	-5407,09
68	-1805,2	-3440,36	-5331,3
69	-1737,05	-3339,12	-5248,65
70	-1667	-3233,08	-5158,96
71	-1595,05	-3122,15	-5062,08
72	-1521,2	-3006,3	-4957,82
73	-1445,45	-2885,47	-4846,01
74	-1367,8	-2759,6	-4726,45
75	-1288,25	-2628,64	-4598,98
76	-1206,8	-2492,54	-4463,41
77	-1123,45	-2351,24	-4319,55
78	-1038,2	-2204,68	-4167,21

Exogenous Value (*m*)	Endogenous Value when μ =1	Endogenous Value when μ =2	Endogenous Value when μ =3
79	-951,05	-2052,82	-4006,2
80	-862	-1895,6	-3836,34
81	-771,05	-1732,96	-3657,42
82	-678,2	-1564,86	-3469,25
83	-583,45	-1391,23	-3271,63
84	-486,8	-1212,02	-3064,36
85	-388,25	-1027,18	-2847,24
86	-287,8	-836,661	-2620,06
87	-185,45	-640,397	-2382,62
88	-81,2	-438,339	-2134,7
89	24,95	-230,433	-1876,1
90	133	-16,625	-1606,6
91	242,95	203,1398	-1326
92	354,8	428,9152	-1034,06
93	468,55	660,7554	-730,576
94	584,2	898,7146	-415,322
95	701,75	1142,847	-88,0757
96	821,2	1393,206	251,3882
97	942,55	1649,847	603,2973
98	1065,8	1912,824	967,881
99	1190,95	2182,19	1345,371
100	1318	2458	1736

These values are shown graphically in figure 6.1, where the thin black curve stands for the simulation given $\mu = 1$, the thin gray curve for $\mu = 2$ and the bold black curve for $\mu = 3$.

Figure 6.1

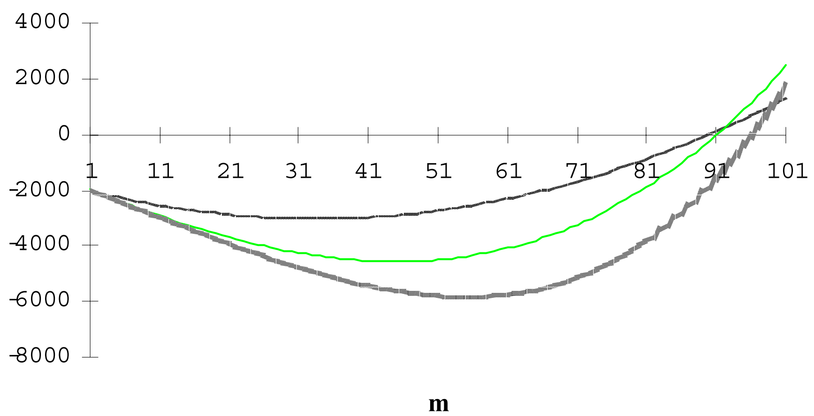

Figure 6.1 shows the values from the table graphically. The object is what value of m_t gives equation (*E*3.17) a zero for different values of μ.

This simulation confirms that the system has a solution (the zero value). It supports following prediction: The higher μ the higher the m_t^*, meaning the slower the flow the greater is the optimal stock of water for each period, *ceteris paribus*. The results can be summed up in the following box.

$$\boxed{m_t^*(\mu = 1) < m_t^*(\mu = 2) < m_t^*(\mu = 3)}$$

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